Phase transition in the Takayasu model with desorption

Satya N. Majumdar,[1] Supriya Krishnamurthy,[2] and Mustansir Barma[1]
[1]*Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India*
[2]*Department of Theoretical Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP, United Kingdom*
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We study a lattice model where particles carrying different masses diffuse and coalesce upon contact, and also unit masses adsorb to a site with rate q or desorb from a site with nonzero mass with rate p. In the limit $p=0$ (without desorption), our model reduces to the well studied Takayasu model where the steady-state single site mass distribution has a power-law tail $P(m)\sim m^{-\tau}$ for large mass. We show that varying the desorption rate p induces a nonequilibrium phase transition in all dimensions. For fixed q, there is a critical $p_c(q)$ such that if $p<p_c(q)$, the steady-state mass distribution, $P(m)\sim m^{-\tau}$ for large m as in the Takayasu case. For $p=p_c(q)$, we find $P(m)\sim m^{-\tau_c}$ where τ_c is a new exponent, while for $p>p_c(q)$, $P(m)\sim \exp(-m/m^*)$ for large m. The model is studied analytically within a mean-field theory and numerically in one dimension.

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I. INTRODUCTION

Many systems in nature, ranging from reaction-diffusion systems to fluctuating interfaces, exhibit nonequilibrium steady states with a wide variety of phases. Of particular interest are the self-organized critical systems where different physical quantities have power-law distributions in the steady state over a wide region of the parameter space [1]. Self-organized criticality has been studied in a variety of model systems ranging from sandpiles to earthquakes. A particularly simple lattice model due to Takayasu, where masses diffuse, aggregate upon contact, and adsorb unit masses from outside at a constant rate, was shown to exhibit self-organized criticality [2]: the steady-state mass distribution has a nontrivial power-law decay for large mass in all dimensions [2]. This model initially generated a lot of attention as it was a simple exactly solvable model of self-organized criticality with close connections [3] to other solvable models such as the Scheidegger river model [4], the voter model [5], and the directed Abelian sandpile model [6]. Recently there has been a renewed interest in this model as simple variants of the Takayasu model have been found useful in modeling the dynamics of a variety of systems, including force fluctuations in granular systems such as bead packs [7], river networks [8], voting systems [9,10], wealth distributions [11], size distributions of fish schools [12], inelastic collisions in granular gases [14], the generalized Hammersley process [13], particle systems in one dimension [15], and various generalized mass transport models [16].

In the Takayasu model, each site of a lattice has a nonnegative mass variable. Starting from an initial random distribution of masses, each mass hops to a nearest neighbor site (chosen at random) and aggregates with the mass there with rate 1. In addition, a unit mass is adsorbed at every site with rate q. While the first move tends to create big masses via diffusion and aggregation, the second move replenishes the lower end of the mass spectrum. At large time t, the mass distribution at any site has the scaling behavior, $P(m,t)\sim m^{-\tau}f(m/t^{\delta})$ with $\delta=1/(2-\tau)$ [17,18]. The interesting point is that even though the average mass per site increases linearly with time, $\langle m\rangle\sim t$, the mass distribution $P(m,t)$ approaches a time-independent power law distribution $P(m)\sim m^{-\tau}$ for $t\to\infty$ [since $f(0)\sim O(1)$] for any nonzero adsorption rate q. The exponent τ is independent of q and is known exactly [2], $\tau=4/3$ in one dimension and $\tau=3/2$ within mean-field theory.

The steady-state mass distribution in the Takayasu model has the same power-law decay for any nonzero adsorption rate q and does not undergo any phase transition. In this paper we show that if we introduce an additional process of desorption of unit masses with rate p in the Takayasu model (we call this the in-out model), a rich steady-state phase diagram emerges in the p-q plane. In particular we show that the system undergoes a nonequilibrium phase transition across a phase boundary $p_c(q)$. Nonequilibrium phase transitions between steady states have been studied extensively in recent years in a variety of systems. Examples include, amongst others, active-absorbing phase transitions in reaction diffusion systems [19], roughening transitions in fluctuating interfaces [20], phase transitions in driven diffusive lattice gas models [21], wetting transitions in solid-on-solid models [22], boundary driven transitions in one-dimensional asymmetric exclusion processes [23], and Bose-Einstein-like condensation in models of aggregation and fragmentation [24,25]. However, we show below that the mechanism of the phase transition and the associated critical properties in the in-out model are very different from those of other models mentioned above.

There are quite a few physical systems where our in-out model may find applications. In nature there exist a variety of systems ranging from colloids [26] to polymer gels [27] where the basic constituents of the system diffuse and coalesce upon contact. For example, in a polymer gel the basic constituents are polymers of different sizes which diffuse in a solution and when an m-mer comes in contact with an n-mer, they aggregate to form an $(m+n)$-mer [27]. Similarly during the growth of a thin film on an amorphous substrate (such as bismuth on carbon), clusters or islands of atoms can diffuse as a whole and when two of them come closer they coalesce [28]. A zeroth order approach to model the dynamics of these systems would be to replace each cluster by a point particle (ignoring its shape) carrying a positive mass which indicates its size or number of atoms. When two particles coalesce their masses add up. In addition many of these systems are *open* in the sense that they can exchange basic units with the adjoining environment. For example, during the growth of a film on a substrate, single adatoms may adsorb on the substrate from the outside vapor or desorb into the vapor from the substrate. We attempt to incorporate these processes on a lattice in the in-out model and show that even this simple model has a very rich steady-state phase diagram. We had introduced this model in an earlier publication [24] and some results were briefly mentioned. In this paper we present a detailed analysis of the model.

The paper is organized as follows. In Sec. II, we define the in-out model precisely and summarize the different phases and the transitions between them. In Sec. III we solve the model analytically within mean-field theory. In Sec. IV we present the numerical results in one dimension and discuss a scaling theory which provides scaling relations between different critical exponents. We conclude in Sec. V with a summary and a discussion of open questions.

II. THE IN-OUT MODEL

For simplicity we define the in-out model on a one-dimensional lattice with periodic boundary conditions; generalizations to higher dimensions are straightforward. Each site of a lattice has a non-negative mass variable $m_i \geqslant 0$. Initially each m_i is chosen independently from any well defined distribution. The dynamics proceed as follows. A site i is chosen at random and then one of the following events can occur.

(1) *Adsorption*. With probability $q/(p+q+1)$, a single particle is adsorbed at site i; thus $m_i \to m_i+1$.

(2) *Desorption*. With probability $p/(p+q+1)$, a single particle detaches from and leaves site i; thus $m_i \to m_i-1$ provided $m_i \geqslant 1$.

(3) *Diffusion and aggregation*. With probability $1/(p+q+1)$, the mass m_i at site i moves to a nearest neighbor site [either $(i-1)$ or $(i+1)$] chosen at random. If it moves to a site that already has some particles, then the total mass just adds up; thus $m_i \to 0$ and $m_{i\pm1} \to m_{i\pm1}+m_i$.

If the site chosen is empty, only adsorption can occur with probability $q/(p+q+1)$.

The in-out model has only two parameters p and q. The question we would like to address is this: For given p and q, what is the single site mass distribution $P(m)$ in the steady state? Note that in the limit $p=0$ (i.e., without the desorption process) our model reduces to the Takayasu model mentioned in the Introduction.

While the Takayasu model (zero desorption, $p=0$) does not have a phase transition in the steady-state, we find that introducing a nonzero desorption rate p induces a rich steady-state behavior in the p-q plane. In fact, we find that there is a critical line $p_c(q)$ in the p-q plane. For fixed q, if we increase p from 0, we find that for all $p<p_c(q)$, the steady-state mass distribution has the same large-m behavior as in the Takayasu case, i.e., $P(m) \sim m^{-\tau}$ where the exponent τ is the Takayasu exponent and is independent of q. Thus the Takayasu phase is stable up to p_c. For $p=p_c(q)$, we find the steady-state mass distribution still decays algebraically for large m, $P(m) \sim m^{-\tau_c}$ but with a new critical exponent τ_c that is bigger than the Takayasu exponent τ. For $p>p_c(q)$, we find that $P(m) \sim \exp(-m/m^*)$ for large m, where m^* is a characteristic mass that diverges if one approaches $p_c(q)$ from the $p>p_c(q)$ side. The critical exponent τ_c is the same everywhere on the critical line $p_c(q)$. This phase transition occurs in all spatial dimensions including $d=1$.

It is easy to write down an exact evolution equation for the mean mass $\langle m\rangle(t)$ per site. Since the diffusion and aggregation move does not change the total mass, the only contributions to the time evolution of $\langle m\rangle$ come from the adsorption and desorption processes. It is then evident that

$$\frac{d\langle m\rangle}{dt} = q - ps(t), \tag{1}$$

where $s(t)$ is the probability that a site is occupied by a nonzero mass. The first term on the right-hand side of the above equation clearly indicates the increase in mass per site due to the adsorption of unit mass. The second term quantifies the loss in mass per site due to the desorption of unit mass taking into acount the fact that the desorption can take place from a site only if the site is occupied by a nonzero mass. Let us fix p and vary q. As long as $q<q_c(p)$, it turns out that in the long time limit $t\to\infty$, the two terms on the right-hand side of the above equation cancel each other and the occupation density reaches the asymptotic time-independent value, $s=q/p$. This indicates that the average mass per site, $\langle m\rangle$ becomes a constant in the long time limit. In fact, we show below that in this phase, the steady-state mass distribution $P(m) \sim \exp(-m/m^*)$ for large m with a finite first moment $\langle m\rangle$. We call this phase the ''Exponential'' phase. However, if $q>q_c(p)$, the occupation density reaches a steady-state value s such that $s<q/p$. As a result in the long time limit, the second term on the right-hand side of Eq. (1) fails to cancel the first term and the mean mass per site $\langle m\rangle(t)$ increases linearly with time, $\langle m\rangle \sim (q-ps)t$. However, as we show below, even though the mean mass diverges in this phase as $t\to\infty$, the mass distribution reaches a steady state, $P(m) \sim m^{-\tau}$ for large m where τ is the Takayasu exponent (which is always less than 2 so that the mean mass diverges). Hence we call this entire phase the ''Takayasu'' phase.

III. MEAN-FIELD THEORY

We first analyze the model exactly within the mean-field approximation, ignoring correlations in the occupancy of adjacent sites. In that case we can directly write down equations for $P(m,t)$, the probability that any site has a mass m at time t,

$$\frac{dP(m,t)}{dt} = -(1+p+q+s)P(m,t) + pP(m+1,t) + qP(m-1,t) + P*P, \quad m \geqslant 1 \tag{2}$$

$$\frac{dP(0,t)}{dt} = -(q+s)P(0,t) + pP(1,t) + s(t). \tag{3}$$

Here $P*P=\Sigma_{m'=1}^{m}P(m',t)P(m-m',t)$ is a convolution term that describes the coalescence of two masses and $s(t)=\Sigma_{m=1}P(m,t)$ denotes the probability that a site is occupied by a nonzero mass.

The above equations enumerate the possible ways in which the mass at a site might change. The first term in Eq. (2) is the ''loss'' term that accounts for the probability that a mass m might move as a whole or desorb or adsorb a unit mass, or a mass from the neighboring site might move to the site in consideration. In this last case, the probability of occupation of the neighboring site $s(t)$ multiplies $P(m,t)$ within the mean-field approximation where one neglects the spatial correlations in the occupation probabilities of neighboring sites. The remaining three terms in Eq. (2) are the ''gain'' terms enumerating the number of ways that a site with mass $m'\neq m$ can gain or lose mass to make the final mass m. The second equation Eq. (3) is a similar enumeration of the possibilities for loss and gain of empty sites.

To solve the equations, we compute the generating function, $Q(z,t)=\Sigma_{m=1}^{\infty}P(m,t)z^m$ from Eq. (2) and set $\partial Q/\partial t=0$ in the steady state. We also need to use Eq. (3) to write $P(1,t)$ in terms of $s(t)$. This gives us a quadratic equation for Q in the steady state. Choosing the root that corresponds to $Q(z=0)=0$, we find

$$2zQ(z)=p(z-1)+qz(1-z)+2sz-\sqrt{(z-1)\Delta(z)}, \tag{4}$$

where

$$\Delta(z)=p^2(z-1)+q^2z^2(z-1)-2pqz(z-1)-4qz(z-sp/q). \tag{5}$$

Note that the occupation density s in the above expression of $Q(z)$ is yet to be determined. The steady-state mass distribution $P(m)$ can be formally obtained from $Q(z)$ in Eq. (4) by evaluating the Cauchy integral,

$$P(m)=\frac{1}{2\pi i}\int_{C_o}\frac{Q(z)}{z^{m+1}}dz \tag{6}$$

over a contour C_o encircling the origin in the complex plane. This expression for $P(m)$, however, will contain the yet to be determined unknown quantity s. In fact, determining s is the most nontrivial part of the mean-field calculation as we show below.

In order to extract the large-m behavior of $P(m)$ from Eq. (6), one needs to deform the contour C_o so that it goes around the branch cut singularities of the function $Q(z)$. From Eq. (4), it is evident that such singularities occur at $z=1$ and also at the roots of $\Delta(z)=0$ where $\Delta(z)$ is given by Eq. (5). Since $\Delta(z)$ is a cubic polynomial in z, it has three roots z_1, z_2, and z_3, each of which can be determined in terms of the unknown quantity s.

We now analyze the large-m behavior of $P(m)$ in different regions of the p-q plane. Let us fix the value of p and increase q from 0. A similar analysis can be carried out for fixed q as a function of p. As we increase q from 0, we encounter the following three regimes.

(i) For small q (with a fixed p), we first assume that the mean mass $\langle m\rangle$ reaches a time-independent constant as $t\to\infty$. This assumption will be justified *a posteriori*. Then from Eq. (1) it follows that the occupation density also reaches a steady-state value, $s=q/p$. Substituting this in the expression for $\Delta(z)$ in Eq. (5), the three roots of $\Delta(z)=0$ are $z_1=1$ and $z_{2,3}=(p+2\mp 2\sqrt{p+1})/q$. Then from Eq. (4) it follows that the only branch cut singularities of $Q(z)$ are at z_2 and z_3 with $z_3>z_2>1$ for small q. Therefore the branch cut at z_2 essentially controls the large-m behavior of $P(m)$ when the contour in Eq. (6) is deformed and by analyzing the integral around this cut we find that for large m,

$$P(m)\sim\exp(-m/m^*)/m^{3/2} \tag{7}$$

with $m^*=1/ln z_2$. Since $P(m)$ decays exponentially in this phase, $\langle m\rangle$ is also finite and nonzero thus justifying the assumption made in the beginning. In this phase the unknown function s is therefore exactly given as $s=q/p$. Note, however, that this analysis is valid as long as $z_2>1$ and the characteristic mass m^* diverges as z_2 approaches 1 from above.

(ii) As the value of q is increased (for fixed p) the roots z_2 and z_3 decrease, until at a critical value $q_c(p)$, the value of z_2 just reaches unity. The double root (z_1 and z_2) at $z=1$ of $\Delta(z)$ then leads to a branch cut singularity of order 3/2 in $Q(z)$ in Eq. (4), which in turn implies

$$P(m)\sim m^{-5/2}. \tag{8}$$

This power-law decay characterizes the critical point and the condition $z_2=1$ determines the locus of the critical line in the p-q plane,

$$q_c(p)=p+2-2\sqrt{p+1}. \tag{9}$$

The value of s is given exactly by $s=q_c/p$.

(iii) As q is increased further $[q>q_c(p)]$ for fixed p, the mean mass per site $\langle m\rangle$ does not reach a time-independent value in the steady state, but increases indefinitely with time. Consequently we cannot use the relation $s=q/p$ anymore. However, $P(m)$ reaches a time-independent distribution. So the question is what is the selection principle that determines the unknown function s in this regime?

Note that at $q=q_c(p)$, the two roots z_1 and z_2 of $\Delta(z)=0$ coincided, $z_1=z_2=1$ and $z_3>1$. As q increases further, since we do not know what s is *a priori*, the exact locations of the three roots of $\Delta(z)=0$ in the complex plane are also unknown. However, since $\Delta(z)$ is a polynomial with real coefficients, if z is a root of $\Delta(z)=0$, so must be its complex conjugate z^*. Thus as q increases beyond $q_c(p)$, there are two possibilities. The first possibility is that all the three roots of $\Delta(z)=0$ are real and distinct. But in that case, as q increases slightly beyond $q_c(p)$, at least one of them must become less than 1. This, however, would lead to an exponential growth of $P(m)$ for large m and hence is ruled out. The second and only possibility is that one of the three roots must be real while the other two are complex conjugates of each other, i.e., $\Delta(z)=(z-z_c)(z-z_c^*)(z-z_3)$ where z_3 is real and z_c in general is complex with its real part less than

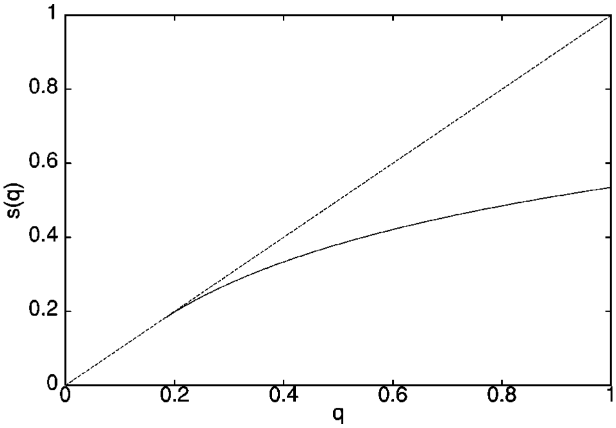

FIG. 1. The function $s(q)$ as a function of q for $p=1$ (shown by the solid line) within the mean-field theory. It deviates from the dotted line $[s(q)=q]$ for $q>q_c=3-2\sqrt{2}$.

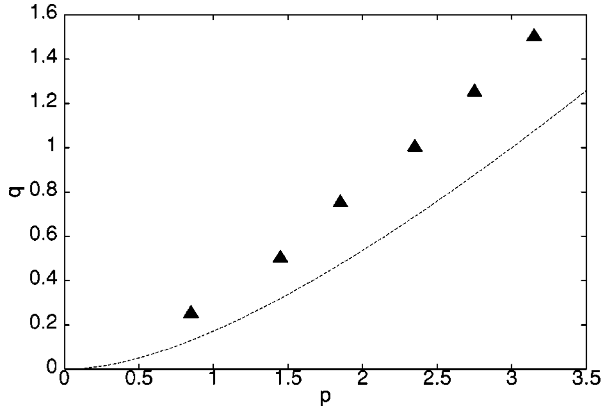

FIG. 2. The phase diagram of the in-out model in the p-q plane. The dotted line denotes the mean-field phase boundary $q_c(p)=p+2-2\sqrt{p+1}$ and the triangles mark the numerically obtained critical points in 1D.

1. However, if the imaginary part of z_c is nonzero, this again can be shown to lead to an exponential divergence of $P(m)$ for large m. Therefore, we are led to the conclusion that z_c must be real and thus $\Delta(z)=0$ must have *double* roots at z_c, i.e., $\Delta(z)=(z-z_c)^2(z-z_3)$ with z_c real. In summary we conclude that for $q>q_c(p)$, z_3 remains greater than 1 and the two roots $z_1=z_2=z_c$ continues to be coincident and real but the common value z_c decreases below 1 as q increases beyond $q>q_c(p)$. This nontrivial ''root sticking'' condition determines the unknown quantity s for $q>q_c(p)$. This condition of double roots can be easily implemented by demanding the two conditions, $\Delta(z_c)=0$ and $\Delta'(z_c)=0$ where $\Delta'=d\Delta(z)/dz$. Also using the relation $\Delta(z)=(z-z_c)^2(z-z_3)$ in Eq. (4), we find that the lowest branch cut singularity of $Q(z)$ is at $z=1$. This order 1/2 singularity then leads to the following asymptotic behavior of $P(m)$:

$$P(m)\sim m^{-3/2}. \tag{10}$$

Thus this entire phase, $q>q_c(p)$, is characterized by the same power-law decay of $P(m)$ as in the mean-field Takayasu model which, as mentioned earlier, corresponds to the zero-desorption $(p=0)$ limit of our model.

As mentioned above, the ''root-sticking'' condition also determines quite nontrivially the occupation density s for $q>q_c(p)$ for fixed p. To determine s explicitly for $q>q_c(p)$ using this condition, let us fix $p=1$ for simplicity even though the calculation can be carried out for any arbitrary p. From Eq. (9), we find $q_c=3-2\sqrt{2}$ for $p=1$. We first substitute the expression for $\Delta(z)$ from Eq. (5) in the ''root-sticking'' conditions, $\Delta(z_c)=0$ and $\Delta'(z_c)=0$. We then eliminate z_c from these equations and find $s(q)$ for $q>3-2\sqrt{2}$ as the only positive root of the cubic equation

$$16s^3-(q^2-12q+24)s^2-(q^3+5q^2+57q+15)s+(q^3+5q^2+39q-2)=0. \tag{11}$$

Thus we can determine the unknown quantity s exactly everywhere in the p-q plane. In Fig. 1, we plot the function $s(q)$ for fixed $p=1$. For $q\leqslant q_c=3-2\sqrt{2}$ we have $s(q)=q$ and for $q>q_c=3-2\sqrt{2}$, $s(q)$ is given by the real positive root of the cubic equation in Eq. (11).

Note that for fixed p, if $q>q_c(p)$, the steady-state value of $s(q)$ (as determined from the ''root-sticking'' conditions) is less than q/p and hence from Eq. (1), we find that the mean mass per site increases linearly with time, $\langle m\rangle\approx vt$ for large t. If one interprets the mass profile as the height of an interface (see Sec. V) then for $q<q_c(p)$, the average ''height'' of the interface becomes a constant as $t\to\infty$, while for $q>q_c(p)$, the average height $\langle m\rangle$ increases linearly with velocity v. The ''velocity'' v, defined more precisely as $v=\lim_{t\to\infty}(\langle m\rangle/t)$, is 0 for $q<q_c(p)$ and nonzero for $q>q_c(p)$. For q slightly bigger than $q_c(p)$, $v\sim[q-q_c(p)]^y$, where y is a critical exponent independent of p. For example, for $p=1$, we find from Eq. (11), $v\approx[q-q_c(1)]^2/(6\sqrt{2}-8)$, indicating that $y=2$ within mean-field theory.

IV. NUMERICAL RESULTS IN ONE DIMENSION AND SCALING THEORY

Having completed the mean-field calculations we now turn to one dimension (1D). While the Takayasu model $(p=0)$ is exactly solvable in $d=1$ [2], the same technique unfortunately does not work for $p>0$. Hence for nonzero p, we had to resort to numerical simulations in $d=1$. The qualitative predictions of mean-field theory, namely the existence of a power-law (Takayasu) phase $[P(m)\sim m^{-\tau_T}]$ and a phase with exponential mass distribution, with a different critical behavior at the transition $[P(m)\sim m^{-\tau_c}]$, are found to hold in 1D as well. Figure 2 shows the results of numerical simulations for the phase diagram along with the mean-field prediction [Eq. (9)] and Fig. 3 displays the numerical data for the decay of the mass distribution $P(m)$ in the two phases and at the transition point. The values obtained, $\tau=4/3$ (same as the exactly solvable $p=0$ case) and $\tau_c\simeq1.833$, are quite different from their mean-field values $\tau=3/2$ and $\tau_c=5/2$, reflecting the effects of correlations between masses at different sites.

If the phase boundary is crossed by increasing q for fixed p, the Takayasu phase is obtained for $q>q_c$. As a function of the small deviation $\tilde{q}\equiv q-q_c$ and large time t, the mass

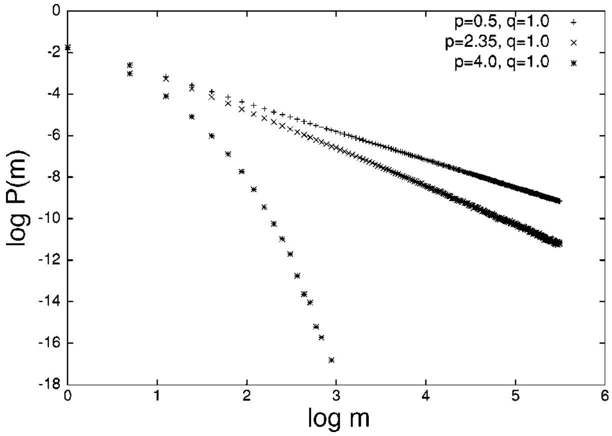

FIG. 3. The steady-state mass distribution $P(m)$ vs m for the in-out model in 1D. The value of q is kept fixed at $q=1$ and the data are shown for three representative values of p, respectively, p is less than, equal to, or greater than $p_c \simeq 2.35$.

distribution $P(m,\tilde{q},t)$ is expected to display a scaling form for large m,

$$P(m,\tilde{q},t) \sim \frac{1}{m^{\tau_c}} Y\left(m\tilde{q}^{\phi}, \frac{m}{t^{\alpha}} \right) \tag{12}$$

in terms of three unknown exponents ϕ, α, τ_c, and the two variable scaling function Y. All other exponents then can be related to these three exponents via scaling relations. We give some examples below.

(a) Consider $\tilde{q}>0$ and $t\to\infty$ limit. Then $P(m,\tilde{q}) \sim (1/m^{\tau_c})Y(m\tilde{q}^{\phi},0)$. But we know that for $\tilde{q}>0$, in the steady state $P(m,\tilde{q}) \sim m^{-\tau}$, where τ is the known Takayasu exponent. This forces the scaling function $Y(x,0)\sim x^{\gamma}$ for large x such that $P(m,\tilde{q}) \sim \tilde{q}^{\phi\gamma}/m^{\tau_c-\gamma}$, indicating $\gamma=\tau_c-\tau$.

(b) Consider again $\tilde{q}>0$ and finite but large t. The mean mass per site, $\langle m\rangle = \int mP(m,\tilde{q},t)dm \sim \tilde{q}^{y}t$, where y is the velocity exponent. Using the scaling form of P, we find $y=\phi[1-\alpha(2-\tau_c)]/\alpha$.

(c) Next we consider the critical point, $\tilde{q}=0$. Using the scaling form, we find that the mean mass $\langle m\rangle \sim t^{\zeta}$ for large t where $\zeta=\alpha(2-\tau_c)$ provided $\tau_c<2$. If $\tau_c>2$ (as in mean-field theory), $\zeta=0$. Also, the root mean square mass fluctuations at the critical point, $\sigma=\sqrt{\langle (m-\langle m\rangle)^2\rangle} \sim \sqrt{\langle m^2\rangle} \sim t^{\beta}$ for large t with $\beta=\alpha(3-\tau_c)/2$. Note that for large t, $\langle m^2\rangle \gg \langle m\rangle^2$ indicating that fluctuations grow faster than the mean as time increases.

Within mean-field theory, by analyzing $P(m)$ explicitly for $\tilde{q}>0$, we find $P(m,\tilde{q}) \sim \tilde{q}/m^{3/2}$ and also $\tau_c=5/2$. From (a) above, this immediately gives $\gamma\phi=1$ and $\gamma=1$, indicating $\phi=1$. Also, we had shown before that the velocity exponent $y=2$ exactly within mean-field theory. Using $y=2$, $\tau_c=5/2$, and $\phi=1$ in (b), we get $\alpha=2/3$. Since $\tau_c=5/2>2$, we note from (c) that $\zeta=0$. Also we find the fluctuation exponent $\beta=1/6$ from the scaling relation in (c). Thus within mean-field theory, we find

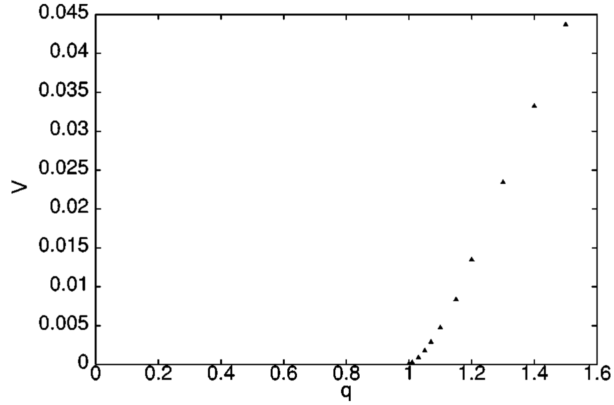

FIG. 4. The velocity v as a function of q for fixed $p=2.35$ in 1D. The velocity is zero for $q<q_c\simeq 1.0$ and increases as $(q-q_c)^y$ for $q>q_c$ with $y\simeq 1.47$ in $d=1$.

$$P(m,\tilde{q},t) \sim \frac{1}{m^{5/2}} Y\left(m\tilde{q}, \frac{m}{t^{2/3}} \right). \tag{13}$$

We have determined the corresponding exponents in $d=1$ numerically. The critical exponent $\tau_c \simeq 1.83$ has already been mentioned (see Fig. 3). In Fig. 4 we plot the velocity v as a function of q for fixed $p=2.35$. The velocity is zero for $q \leqslant q_c \approx 1.0$ and increases as a power law, $v\sim(q-q_c)^y$ for small $\tilde{q}=(q-q_c)$. We find $y\simeq 1.47$. Note that since q_c is not known exactly, this exponent is difficult to determine numerically and is subject to large error bars. We also find that at the critical point $q_c\approx 1$, the mean mass grows as $\langle m\rangle \sim t^{\zeta}$ with $\zeta\simeq 0.12$. Note the difference from the mean-field theory where $\langle m\rangle$ does not grow with time at the critical point ($\zeta=0$). To measure the fluctuations at the critical point, we performed finite-size studies of the time-dependent width $W^2(t,L)=\Sigma_{i=1}^{L}(m_i-\langle m\rangle)^2/L$ at the critical point, where L is the system size. This is expected to obey the scaling form $W\simeq t^{\beta}Z(t/L^z)$; the value of z is expected to be 2 as the movement of masses is diffusive. Figure 5 shows the

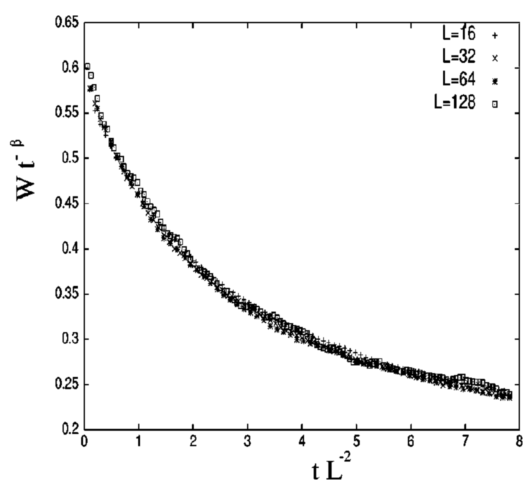

FIG. 5. Scaling plots of the finite-size studies of width versus time for four different system sizes $L=16$, $L=32$, $L=64$, and $L=128$ at the critical point $p_c\approx 2.352$ with fixed $q=1$. The width is expected to follow the scaling form $W\approx t^{\beta}Z(t/L^z)$. The best collapse is obtained for $\beta\simeq 0.358$ with $z=2$.

scaling plot of W/t^{β} versus t/L^{z} for four different system sizes $L=16$, 32, 64, and 128 at the critical point $p\approx 2.35$ for fixed $q=1$. We fix $z=2$ and find the best collapse of data for $\beta \simeq 0.358$. These exponent estimates in $d=1$ are consistent with the scaling relations mentioned in (a)–(c).

V. SUMMARY AND DISCUSSION

In this paper we have studied a simple lattice model where masses diffuse and aggregate with rate 1, unit masses adsorb at any lattice site with rate q, and unit masses desorb from a site (provided the site is occupied by a mass) with rate p. For $p=0$ (without the desorption process), our model reduces to the well studied Takayasu model where the steady-state single site mass distribution has a power-law decay, $P(m)\sim m^{-\tau}$ for large m for any nonzero q. We show that varying the desorption rate p induces a nonequilibrium phase transition at a critical value $p=p_c(q)$. For $p<p_c(q)$, $P(m)\sim m^{-\tau}$ for large m as in the Takayasu ($p=0$) case. For $p=p_c(q)$, $P(m)\sim m^{-\tau_c}$ where τ_c is a new exponent and $P(m)\sim \exp(-m/m^*)$ for $p>p_c(q)$. We have solved the model analytically within the mean-field theory and calculated all the mean-field exponents exactly. In one dimension, we have computed the exponents numerically. We have also presented a general scaling theory.

Our model generalizes the Takayasu model and exhibits a nontrivial phase transition. There was an earlier generalization [29] of the model where instead of carrying positive masses, the diffusing particles carried charges Q of either sign while a random charge I, drawn from an arbitrary distribution, was added with rate q to a lattice site. In this ''charge'' model, the steady-state single site charge distribution $P(Q)$ was found [29] to have a power-law tail (as in the mass case), $P(Q)\sim Q^{-\tau}$ for large positive Q when the mean charge injected was positive, $\langle I\rangle>0$ whereas for $\langle I\rangle=0$, $P(Q)\sim Q^{-\tau_1}$ for large positive Q. It was shown that the exponent $\tau_1=5/3$ in $d=1$ and $\tau_1=2$ within mean-field theory [29]. Though this change of exponent at a critical value $\langle I\rangle=0$ is similar to that in our model qualitatively, the exponent τ_c of the in-out model is very different from that of the charge model. This difference can be traced back to the mass positivity constraint in the in-out model, i.e., the desorption of a unit mass can take place from a lattice site only if the site has a nonzero mass.

In the in-out model, the total mass is not conserved due to the moves involving adsorption and desorption of unit mass. It is interesting to ask what would happen if the desorption of a unit mass from a site were followed by adsorption at a neighboring site, so that the total mass would be conserved in every move. This was investigated using a lattice model [24] and earlier within a rate equation approach [25]. In this conserved-mass model there is also a phase transition, but of a different character. It was found that there is an exponential phase (at a high desorption-adsorption rate), separated by a critical line from a phase with a power-law mass distribution $P(m)\sim m^{-\tau_{\text{conserved}}}$. This distribution coexists with an infinite aggregate which accommodates a finite fraction of the total mass—a real space analog of Bose-Einstein condensation [24]. The exponent $\tau_{\text{conserved}}$ was found to be 5/2 within mean-field theory [24,25] and $\simeq 2.33$ in 1D [24], and the same exponent was found to describe $P(m)$ at the critical point. Evidently, the lack of mass conservation in the in-out model is responsible for the absence of the infinite aggregate in its high q phase, as well as the change in the power to τ in the Takayasu phase and τ_c at criticality.

Another interesting difference between the conserved and the in-out model is the effect of a preferred direction for the motion of masses (a mass at site i hops with a higher probability to $i-1$ than $i+1$). We have checked that such a bias does not change the critical exponents of the in-out model. However, for the conserved mass model, the bias in direction changes the value of the exponents at the transition and in the aggregate phase [24].

The phase transition in the in-out model has some interesting implications for nonequilibrium wetting transitions if we interpret the configuration of masses as an interface profile regarding m_i as a local height variable. Although the dynamics of the mass profile in our in-out model is not physical when interpreted as interface dynamics, nevertheless the phase transition in our model can be qualitatively interpreted as a nonequilibrium wetting transition of the interface. In the in-out model, the fact that the mass at each site is necessarily non-negative translates into the restriction that the mass profile is always above a wall at a fixed height (in our case 0). The presence of this constraint is the key factor for the wetting transition. At fixed p, as we increase q, the mean height $\langle m\rangle$ does not grow with time as long as $q<q_c$. This is our ''Exponential'' phase where the mass or the interface profile is bound to the substrate at zero height. This phase is also ''smooth'' as the mean square height fluctuation does not grow with system size. For $q>q_c$, the interface unbinds from the substrate and the mean height $\langle m\rangle \sim vt$ grows linearly with time with a velocity v. This phenomenon is similar to the ''wetting'' or ''depinning'' of interfaces in general. In this ''wet'' phase (the Takayasu phase in our model), the interface is rough. Unlike recently studied models of nonequilibrium wetting, where the interface in the growing phase is self-affine [22,30], our model describes a much rougher interface for $q>q_c$. At the transition $q=q_c$, though, the interface is self-affine with a roughness exponent $\chi=z\beta\simeq 0.7$.

There are various open questions that remain to be settled. In this paper we have only studied the phase transition in the steady-state single site mass distribution function. It would be very interesting to study the spatial correlations between masses at different sites and to track the behavior of a mass-mass correlation function as one crosses the phase boundary in the p-q plane.

Also in this paper we have only studied the simplest model where the rates of adsorption, desorption, and hopping are constants and independent of particle mass. An important question is whether this phase transition would persist for general mass-dependent rates. In earlier work [31] a model with aggregation, adsorption, and desorption was studied, but no transition to a power-law phase was found; the difference is traceable to the fact that in that model, the rate of removal of mass is proportional to the mass, unlike the unit-mass desorption process considered in the in-out model. It is therefore highly desirable to identify the class of models with mass-dependent rates where the phase transition described here will persist.

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