Persistence with Partial Survival

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We introduce a parameter p called partial survival in the persistence of stochastic processes and show that for smooth processes the persistence exponent $\theta(p)$ changes continuously with p, $\theta(0)$ being the usual persistence exponent. We compute $\theta(p)$ exactly for a one-dimensional deterministic coarsening model, and approximately for the diffusion equation. Finally we develop an exact, systematic series expansion for $\theta(p)$, in powers of $\epsilon = 1 - p$, for a general Gaussian process with finite density of zero crossings. [S0031-9007(98)07216-0]

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Recently considerable theoretical and experimental effort has been devoted to understanding first-passage statistics in nonequilibrium systems. These include the Ising, Potts, and time-dependent deterministic Ginzburg-Landau (TDGL) models undergoing zero-temperature phase-ordering dynamics [1–4], the diffusion equation with random initial conditions [5,6], the global magnetization undergoing critical dynamics [7], several reaction-diffusion systems [8], fluctuating interfaces [9], and a randomly accelerated particle [10]. Typically one is interested in "persistence," i.e., the probability $P_0(t)$ that, at a fixed point in space, the stochastic process (such as an Ising spin or the diffusion field) does not change sign up to time t. In the examples mentioned previously, this probability decays as a power for large time t, $P_0(t) \sim t^{-\theta}$, where the persistence exponent θ is nontrivial due to the non-Markovian nature of the process in time *at a fixed point in space*. This exponent has recently been measured experimentally in a 2D liquid crystal system [11], and also for 2D soap froth [12] and breath figures [13]. The theoretical computation of θ, however, despite a few exact and approximate results, remains a major challenge.

Even for the simple diffusion equation, $\partial_t \phi = \nabla^2 \phi$ starting from random initial configuration, the exponent θ is known only numerically and within an independent interval approximation (IIA) [5], though there is a recent conjecture [14] for an exact θ that remains to be proved. The IIA result, though in excellent agreement with numerical simulations, is hard to improve systematically. The central result of this Letter is to derive a systematic series expansion for θ in terms of a suitable expansion parameter. This expansion is *exact* order by order and when truncated at second order already gives good results for the diffusion equation. But this exact series expansion technique is more general and goes beyond the diffusion equation. We show that it can be applied to compute the persistence exponent, order by order, for a wide class of stochastic processes which includes the diffusion equation, random acceleration, and the 1D TDGL model as special cases.

Our result is also useful for a related problem which has wide applications in diverse fields ranging from information theory to stock markets and oceanography. Consider a stochastic Gaussian *stationary* process $X(T)$ characterized completely by its two-time correlator, $\langle X(0)X(T)\rangle = f(T)$. The process $X(T)$ can be used to model, e.g., the current in an electrical circuit or the price of a stock. Given $f(T)$, what is the probability $P_0(T)$ that the signal stays above (or below) a certain level, say zero, up to time T? This problem has been studied for many years [15,16], and it is known that if $|f(T)| < 1/T$ for large T, then $P_0(T) \sim \exp(-\theta T)$ for large T [15]. The exponent θ depends quite sensitively on the full function $f(T)$ and is very hard to compute for general $f(T)$ [4]. For a Markov process, where $f(T) = \exp(-\lambda T)$ for all T, it is known that $\theta = \lambda$ [15]. However, for non-Markov processes, where $f(T)$ is not a pure exponential, very little is known. Only recently a perturbation theory result for θ was developed for processes close to Markovian [4,17].

The persistence problem for the diffusion equation in d dimensions can be exactly mapped to a Gaussian stationary process, with $f(T) = [\mathrm{sech}(T/2)]^{d/2}$, by identifying $T = \ln t$ and $X(T) = \phi(x,t)/\sqrt{\langle \phi^2(x,t)\rangle}$ [5]. The probability of no zero crossing then decays as $P_0(T) \sim \exp(-\theta T) = t^{-\theta}$. The series expansion technique that we develop below can be used to compute θ for arbitrary $f(T)$ as long as $f(T) \sim 1 - aT^2 + \ldots$ for small T. Such Gaussian processes are called *smooth* as they have a finite density of zero crossings, $\rho = \sqrt{-f''(0)}/\pi$ [18].

The key strategy underlying our technique is to first generalize the usual persistence problem by introducing a partial survival factor p as follows. The usual persistence, say, in the diffusion equation, is the fraction of points in space where the diffusion field has not changed sign even once up to time t. One way to compute this is to start with a random initial configuration of the field and put a particle at each point in space to act as a counter. At subsequent times, whenever the field changes sign at any point, the particle there dies. The persistence is simply the fraction of particles still surviving at time t. We now

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generalize this by assigning the rule that whenever the field changes sign at a point, the particle there survives with probability p and dies with probability $(1 - p)$. We then compute the fraction of particles, $P(p,t)$ left after time t. Thus, $p = 0$ corresponds to usual persistence $P(0,t)$. A somewhat similar generalization was recently studied in the context of "adaptive persistence" problems [19].

This generalization has several implications. It is easy to see that if $P_n(t)$ denotes the probability of n zero crossings in time t of the underlying single site process, then $P(p,t)$ is simply the generating function,

$$P(p,t) = \sum_{n=0}^{\infty} p^n P_n(t). \tag{1}$$

For $p = 0$, $P(0,t) = P_0(t)$, the usual persistence, decaying for large t as $t^{-\theta(0)}$. In the other limit, $p = 1$, the particles always survive: $P(1,t) = 1$, implying $\theta(1) = 0$. It is interesting to analytically continue Eq. (1) to negative p. In fact, for $p = -1$ this is simply the autocorrelation function, $P(-1,t) = A(t) = \langle \mathrm{sgn}[\phi(x,0)]\,\mathrm{sgn}[\phi(x,t)]\rangle$, which decays as $t^{-\lambda/2}$, where λ is a well-studied exponent in phase-ordering systems [20]. In fact, we show below that for smooth processes, $P(p,t) \sim t^{-\theta(p)}$ for large t where the exponent $\theta(p)$ depends continuously on p as p varies from -1 to $+1$. Moreover, the quantity $A_p(t) = P(-p,t)/P(p,t)$ is just the autocorrelation function averaged only over points with surviving particles, when the survival probability is p. So if $A_p(t) \sim t^{-\lambda_p}$, we have $\lambda_p = \theta(-p) - \theta(p)$. This generalization thus puts both the autocorrelation and the persistence exponents as members of a wider family of exponents.

We first establish the continuous dependence of $\theta(p)$ on p for smooth processes by computing $\theta(p)$ exactly for a non-Gaussian process, namely, the 1D deterministic TDGL model, and then approximately within IIA for the diffusion equation. We then proceed to compute $\theta(p)$ for any smooth Gaussian stationary process by expanding around $p = 1$. This series expansion result for $\theta(p)$ in powers of $\epsilon = 1 - p$ is exact order by order.

If a system, such as the Ising model, is quenched from a high-temperature disordered phase to zero temperature, domains of "up" and "down" phases form and grow with time. The evolution of the order-parameter field ϕ can be modeled by the deterministic TDGL equation, $\partial_t \phi = \nabla^2 \phi - V'(\phi)$, where $V(\phi)$ is a symmetric double well potential with minima at $\phi = \pm 1$. In 1D, at late times the system breaks up into alternate up and down domains and coarsens by successively eliminating the boundaries of the smallest domain, i.e., by flipping the signs of ϕ simultaneously at all points inside the smallest domain [21]. The density of persistent, or "dry" parts where ϕ has not changed sign then scales as $\sim \langle l \rangle^{-\theta(0)}$ where $\langle l \rangle$ is the average length of growing domains, which serves as "time" in this problem. The exponent $\theta(0)$ was computed exactly by noting that the dynamics does not generate correlations between neighboring domains [3]. We now introduce the partial survival factor p in this dynamics.

We start with a random distribution of intervals or domains and assign a particle to each point in space. The dynamics merges the smallest interval $I_{\min}$ with its two neighbors I_1 and I_2 to make one single interval I. The lengths $l(I)$ and the dry part $d(I)$ (i.e., the number of live particles in the interval I) evolve as $l(I) = l(I_1) + l(I_2) + l(I_{\min})$ and $d(I) = d(I_1) + d(I_2) + p d(I_{\min})$. Thus the only difference from the calculation in Ref. [3] is the p-dependent term in the dry part. The rest of the calculation is similar to that in Ref. [3], and we just outline the method without details. One writes down the evolution equations for the number of intervals of length l and the average dry part carried by such an interval, and one solves exactly for the associated scaling functions by taking Laplace transforms. Demanding that the first moments of these scaling functions are finite gives a transcendental equation for $\theta(p)$,

$$\int_0^\infty dt e^{-t} t^{-1-\theta}[(1-p)(1-t-e^{-t})e^{r(t)} + 2\theta(1+p)t + \theta(1-p)t^2 e^{-r(t)}] = 0, \tag{2}$$

where $r(t) = -\gamma - \sum_{n=1}^{\infty}(-t)^n/n\,n!$, γ being Euler's constant. Clearly, for $p = 1$ one gets $\theta(1) = 0$ from the above equation as expected. For $p = 0$, it reduces to the equation for $\theta(0)$ as obtained in Ref. [3]. For $p = -1$, one recovers the equation for $\theta(-1) = \lambda$ of Ref. [22]. Figure 1 shows $\theta(p)$ as a function of p for $-1 \le p \le 1$, obtained by numerically solving Eq. (2).

We now turn to the diffusion equation, $\partial_t \phi = \nabla^2 \phi$, starting from a random initial configuration. We first carry out a numerical simulation to compute $P(p,t)$ for finite p following the procedures of Ref. [5]. Figure 2 shows the asymptotic decay of $P(p,t)$ with t on a log-log plot for $p = 0$ and $p = 0.5$ in 1D. Clearly the exponents are quite different. For example, for $p = 0$, $\theta(0) = 0.1207 \pm 0.0005$ as in Ref. [5], but for $p = 0.5$,

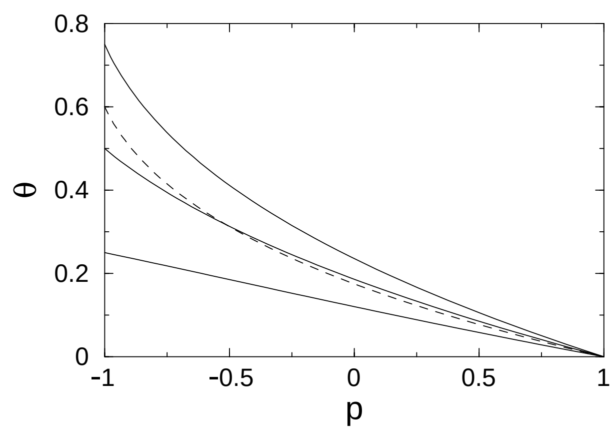

FIG. 1. Dashed line: The exponent $\theta(p)$ for the 1D TDGL model, obtained from Eq. (2). Solid lines: The IIA estimates for $\theta(p)$ for the diffusion equation in (bottom to top) 1, 2, and 3 dimensions.

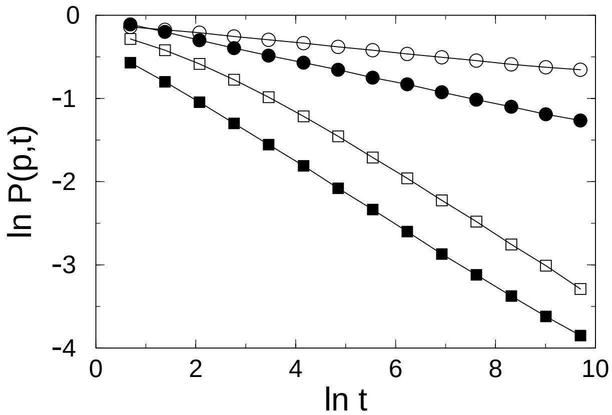

FIG. 2. Time dependence of the generalized persistence probability, $P(p,t)$, from numerical simulations. Circles: The 1D diffusion equation for $p = 0$ (filled) and $p = 0.5$ (open). Squares: The 1D Glauber model at zero temperature for $p = 0$ (filled) and $p = 0.5$ (open). The lines are guides to the eye.

we find $\theta(1/2) = 0.0588 \pm 0.0005$. Unfortunately we have not been able to compute $\theta(p)$ exactly. However, the IIA used in Ref. [5] to compute $\theta(0)$ can be easily extended to compute $\theta(p)$ very accurately for all p as we now show.

Consider the normalized process $X(T) = \phi/\langle\phi^2\rangle$ as a function of $T = \ln t$. The zero-crossing events of ϕ are the same as those of X, but $X(T)$ is a *stationary* Gaussian process characterized completely by its two-time correlator, $f(T) = \langle X(0)X(T)\rangle = [\mathrm{sech}(T/2)]^{d/2}$. With a nonzero survival factor p, the fraction of live particles after time T is then $P(p,T) = \sum_{n=0}^{\infty} p^n P_n(T)$ and will decay at late times as $\exp[-\theta(p)T] = t^{-\theta(p)}$. To evaluate $P_n(T)$, the probability of n zero crossings by X in time T, we note that the Laplace transforms, $\tilde{P}_n(s) = \int_0^\infty dT \exp(-sT)P_n(T)$, were evaluated in Ref. [5] using IIA, i.e., assuming that successive intervals between zero crossings of X are statistically independent. Using these results from Ref. [5], and carrying out the sum over n, gives $\tilde{P}(p,s)$. Since $P(p,T) \sim \exp[-\theta(p)T]$ for large T, $P(p,s)$ will have a simple pole at $s = -\theta(p)$. Using this, we finally get $\theta(p)$ as a solution of the equation

$$\frac{1-p}{1+p} = \theta\pi\sqrt{\frac{2}{d}}\left\{1 + \frac{2\theta}{\pi}\int_0^\infty dT \exp(\theta T) \times \sin^{-1}[\mathrm{sech}^{d/2}(T/2)]\right\}. \quad (3)$$

The solution is plotted in Fig. 1 for $d = 1,2,3$. We note in the two extreme limits, $p = 1$ and $p = -1$, the IIA gives $\theta(1) = 0$ and $\theta(-1) = d/4$, which are exact. For intermediate values of p, the IIA results are in excellent agreement with numerical simulations. For example, for $p = 1/2$ the IIA gives $\theta_{\mathrm{IIA}} = 0.058\,230\,44\ldots$, compared to $\theta_{\mathrm{sim}} = 0.0588 \pm 0.0005$.

Having established the continuous p dependence of $\theta(p)$ for two smooth processes, we now derive an exact series expansion of $\theta(p)$ near $p = 1$ for a general smooth, Gaussian, stationary process $X(T)$, characterized by its two-time correlator $f(T)$. The basic idea is straightforward. We start with the definition (1) of $P(p,t)$ as a generating function. Writing $p^n = \exp(n \ln p)$ and expanding the exponential, we obtain an expansion in terms of the moments of n, the number of zero crossings:

$$\ln P(p,T) = \sum_{r=1}^{\infty} \frac{(\ln p)^r}{r!}\langle n^r\rangle_c\,, \quad (4)$$

where $\langle n^r\rangle_c$ are the cumulants of the moments. Using $p = 1 - \epsilon$, we express the right-hand side as a series in powers of ϵ. Since $P(p,T)$ is expected to decay for large T as $\exp[-\theta(p)T]$, we obtain a series expansion of $\theta(p)$ by taking the limit

$$\theta(p) = -\lim_{T\to\infty}\frac{1}{T}\ln P(p,T) = \sum_{r=1}^{\infty} a_r\epsilon^r. \quad (5)$$

The coefficients a_r's involve the cumulants.

Fortunately the computation of the moments of n is relatively straightforward, though tedious for higher moments. For example, the first moment $\langle n\rangle$, i.e., the expected number of zero crossings in time T, was computed by Rice [18]: $\langle n\rangle = T\sqrt{-f''(0)}/\pi$, implying $a_1 = \sqrt{-f''(0)}/\pi$. The second moment, $\langle n^2\rangle$, was computed by Bendat [23]. Using this result and after some algebra we have computed the coefficient a_2, which already looks complicated. We just quote the final result here (details will be published elsewhere [24]):

$$a_2 = \frac{1}{\pi^2}\int_0^\infty [S(\infty) - S(T)]dT\,, \quad (6)$$

where $S(T)$ is given by

$$S(T) = \frac{\sqrt{M_{22}^2 - M_{24}^2}}{[1 - f^2(T)]^{3/2}}[1 + H\tan^{-1}H], \quad (7)$$

with $H = M_{24}/\sqrt{M_{22}^2 - M_{24}^2}$. The M_{ij}'s are the cofactors of the 4×4 symmetric correlation matrix C between 4 Gaussian variables $[X(0), \dot{X}(0), X(T), \dot{X}(T)]$. The elements of C can easily be computed from the correlator $f(T)$. For example, $C_{11} = \langle X(0)X(0)\rangle = f(0)$, $C_{14} = \langle X(0)\dot{X}(T)\rangle = f'(T)$, $C_{24} = \langle\dot{X}(0)\dot{X}(T)\rangle = -f''(T)$, and so on.

Although these expressions look complicated, in many cases the function $S(T)$ can be evaluated explicitly and the integral for a_2 can be performed analytically. For example, for 2D diffusion equation, where $f(T) = \mathrm{sech}(T/2)$, we get

$$\theta(p = 1 - \epsilon) = \frac{1}{2\pi}\epsilon + \left(\frac{1}{\pi^2} - \frac{1}{4\pi}\right)\epsilon^2 + O(\epsilon^3). \quad (8)$$

Keeping terms up to second order and putting $\epsilon = 1$ (in the same spirit as ϵ expansion in critical phenomena) gives $\theta(0) = (\pi + 4)/4\pi^2 = 0.180\,899\ldots$, just 3.5% below the simulation value, $\theta_{\rm sim} = 0.1875 \pm 0.0010$ [5]. Note that though the IIA estimate, $\theta_{\rm IIA} = 0.1862$, is even closer to the simulation, it cannot be improved systematically. The series expansion estimate, on the other hand, can be improved systematically order by order.

We have also computed, for the first time, the third moment $\langle n^3 \rangle$ for a general smooth correlator $f(T)$. We then use this to compute a_3. The expressions involve the elements of a 6×6 correlation matrix and are not particularly illuminating [24], so we skip the details here. As an example, we computed the series up to third order for the random acceleration process, $d^2x/dt^2 = \eta$ (η is a Gaussian white noise) which can be transformed to a Gaussian stationary process with $f(T) = [3\exp(-T/2) - \exp(-3T/2)]/2$ [5]. We find

$$\theta(p) = \frac{\sqrt{3}}{2\pi}\left(\epsilon - \frac{1}{6}\epsilon^2 + \frac{11}{72}\epsilon^3 + O(\epsilon^4)\right). \qquad (9)$$

Putting $\epsilon = 1$, we get to third order, $\theta(0) = 0.271\,835\,775\ldots$, which should be compared to its exact value 0.25 [10]. We note that the series oscillates around the exact value 0.25 as the order increases.

We note that the series expansion will fail for nonsmooth Gaussian processes, whose moments of zero crossings are not finite. As an example, consider ordinary Brownian motion, $dx/dt = \eta$, which can be mapped to a stationary Gaussian-Markov process with correlator $f(T) = \exp(-T/2)$ using the change of variables discussed before. For this process it is well known [15] that the moments of zero crossings are infinite: if the process crosses zero once, then it crosses again infinitely many times immediately afterwards [25]. Thus only the $n = 0$ term contributes to the sum (1), giving $P(p,T) \approx P(0,T) \sim \exp(-T/2) = t^{-1/2}$ for large t. Thus $\theta(p) = 1/2$ for all $0 \le p < 1$, except at $p = 1$ where $\theta(1) = 0$. Since $\theta(p)$ is discontinuous at $p = 1$, no expansion around $p = 1$ is possible.

The same conclusion holds for the $T = 0$ Glauber dynamics of the Ising model. In this case, the usual persistence exponent $\theta(0)$ was recently computed exactly in 1D [2] and approximately in higher dimensions [4]. The exact value in 1D is $\theta(0) = 3/8$ [2]. Even though the spin $S_i(t)$ at a given site i is no longer a Gaussian process, it is nonsmooth nevertheless; i.e., if a spin flips once, it usually flips many times immediately afterwards. This fact can be tested easily by computing the exponent $\theta(p)$ for nonzero p. In Fig. 2 we show the asymptotic dependence of $P(p,t)$ on t on a log-log plot for $p = 0$ and $p = 1/2$ for the 1D $T = 0$ Glauber model. In contrast to the diffusion case, the asymptotic slopes are the same and given by 0.375 ± 0.002. We have checked this fact for other values of p, and conclude that $\theta(p)$ is independent of p for $0 \le p < 1$ [26], while clearly $\theta(1) = 0$. Thus the p dependence of $\theta(p)$ provides important information about the nature of the smoothness of the underlying stochastic process.

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