Slow relaxation in a constrained Ising spin chain: Toy model for granular compaction

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We present detailed analytical studies on the zero-temperature coarsening dynamics in an Ising spin chain in the presence of a dynamically induced field that favors locally the "−" phase compared to the "+" phase. We show that the presence of such a local kinetic bias drives the system into a late time state with average magnetization m equal to -1. However the magnetization relaxes into this final value extremely slowly in an inverse logarithmic fashion. We further map this spin model exactly onto a simple lattice model of granular compaction that includes the minimal microscopic moves needed for compaction. This toy model then predicts analytically an inverse logarithmic law for the growth of density of granular particles, as seen in recent experiments and thereby provides a mechanism for the inverse logarithmic relaxation. Our analysis utilizes an independent interval approximation for the particle and the hole clusters and is argued to be exact at late times (supported also by numerical simulations).

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I. INTRODUCTION

Slow relaxation dynamics naturally occurs in systems with *quenched disorder* such as spin glasses and has remained a subject of long-standing interest [1]. However, systems without quenched disorder such as structural glasses also exhibit slow dynamics. It is believed that the slow relaxation in the latter systems is due to *kinetic disorder*, induced by the dynamics itself [2]. Another important class of systems without quenched disorder is granular material, where once again kinetic disorders are responsible for slow relaxation. In a recent experiment [3], a cylinder packed loosely with glass beads was tapped mechanically and it was found that the system gets more and more compact with time. However, the density $\rho(t)$ compactified rather slowly with time as, $\rho(\infty)-\rho(t)\sim 1/\ln(t)$. How robust is this inverse logarithmic relaxation? Is it only specific to granular systems or does this also occur in other out-of-equilibrium systems in the presence of kinetic disorders?

In this paper we study, in detail, the effect of kinetic disorders on the relaxation dynamics of an Ising system quenched from a high-temperature disordered phase into a low-temperature ordered phase. In the absence of kinetic disorders, the dynamics of such a system is well understood [4]. As time progresses, domains of equilibrium low-temperature ordered phases (consisting predominantly of up and down spins, respectively) form and grow. The average linear size of a domain grows with time as $l(t)\sim t^{1/2}$ for zero-temperature nonconserved dissipative dynamics. How does disorder, *quenched* or *kinetic*, affect this simple dynamics? The effect of quenched ferromagnetic disorder on this phase ordering kinetics has been studied extensively [4]. Essentially the quenched disorder tends to pin the domain walls, leading to complete freezing at $T=0$. However, at small nonzero temperatures, the domains still coarsen via activated dynamics but extremely slowly as $l(t)\sim(\ln t)^{1/4}$ [4]. The purpose of this paper is to explore the effect of *kinetic disorder* on the phase ordering dynamics. A shorter version of this paper with the main analytical results and their numerical confirmation has appeared elsewhere [5]. Here we explore the dynamics in more detail, and also elaborate its connection to other systems such as granular material and reaction-diffusion systems.

In this paper, we study the zero-temperature dynamics of an Ising chain in the presence of a specific type of kinetic disorder, namely, a dynamically induced magnetic field. If a small uniform external field (say in the down direction) is put on in an Ising system following a rapid quench from infinite temperature to $T=0$, the system rapidly relaxes into a pure state with all spins down. In this case, the symmetry between the two ordered pure states is broken globally. Interesting physics occurs when, instead of a global external bias, the symmetry between the pure states is broken locally by the dynamics itself. In this paper, we investigate the effect of this particular kinetic disorder and show that this system also gives rise to a very slow dynamics. In particular, it gives rise to inverse logarithmic relaxation (ILR) of magnetization, very similar to the density compaction in granular systems. Our study therefore suggests that the ILR is a very robust phenomenon and is not just limited to granular systems or specific types of kinetic disorders.

The paper is organized as follows. In Sec. II, we define our model precisely and summarize the main results. In Sec. III, we establish the connection between our spin model and a lattice model of granular compaction. We also show that our model can be viewed as a one-dimensional (1D) reaction-diffusion model when the dynamics is described in terms of the kinks between domains of opposite phases. In Sec. IV, we derive some exact results. In Sec. V, we analyze the dynamics via an independent interval approximation (IIA). In Sec. VI, we argue that the IIA results become exact in the limit when the initial volume fraction of one of the phases is small. In Sec. VII, we extend this argument to other volume fractions as well. In Sec. VIII, we present a heuristic approach to support the IIA results. Finally we conclude in Sec. IX.

II. THE MODEL AND THE OVERVIEW OF RESULTS

We consider a simple Ising spin chain with spins $S_i=\pm 1$. Starting from a given initial configuration the system

evolves by single spin flip continuous time dynamics. The rate of flipping of a given spin depends on the its neighboring spins. We denote the rate of spin flip $S_i \to -S_i$ by $W(S_i;S_{i-1},S_{i+1})$ where S_{i-1} and S_{i+1} are the two neighboring spins. In our model the rates are specified as follows:

$$W(+;++)=W(-;--)=0,$$

$$W(+;-+)=W(+;+-)=W(-;+-)=W(-;-+)=\tfrac{1}{2},$$

$$W(+;--)=1,$$

$$W(-;++)=\alpha. \tag{1}$$

Note that the case $\alpha=1$ corresponds to the usual zero-temperature Glauber dynamics [6] that preserves the symmetry between the up and down phases. However, if $\alpha<1$, the flipping of a down-spin sandwiched between two up-spins is not as likely as the flipping of an up-spin sandwiched between two down-spins. Thus $\alpha<1$ clearly breaks the symmetry between the up and down phases. However this symmetry is broken only dynamically, i.e., not everywhere but only at the location of the triplets $(+,-,+)$. Thus the isolated "$-$" spins (surrounded on both sides by a "$+$") tend to block the coalescence of "$+$" domains and locally favor the "$-$" spins. One can argue that the asymptotic dynamics at late times is similar for any $\alpha<1$. In other words, $\alpha=0$ is an attractive fixed point. We therefore restrict ourselves only to the case $\alpha=0$.

To see the effect of the local dynamical constraint more precisely, we derive (following Glauber's calculation for $\alpha=1$ [6]) the exact evolution equation for the magnetization $m(t)=\langle S_i\rangle$ for the $\alpha=0$ case:

$$\frac{d}{dt}\langle S_i\rangle=-2P(1,-1,1), \tag{2}$$

where $P(\sigma_{i-1},\sigma_i,\sigma_{i+1})(t)=\langle(1+\sigma_{i-1}S_{i-1})(1+\sigma_iS_i)(1+\sigma_{i+1}S_{i+1})\rangle/8$ is the three-point probability to find the sequence of spins $(\sigma_{i-1},\sigma_i,\sigma_{i+1})$ about the site i and we have used translational invariance. Hence $P(1,-1,1)(t)$ denotes the probability of the occurrence of the triplet "$+-+$" at time t. We note that for the case $\alpha=1$, $d\langle S_i\rangle/dt=0$ [6], indicating that the magnetization does not evolve with time. In our case, due to the triplet defects "$+-+$," the average magnetization decays with time. If $L_\pm(t)$ denote the fractions of "$+$" and "$-$" spins, then using $L_\pm=(1\pm m)/2$ we find from Eq. (2), $dL_\pm/dt=\mp R_1(t)$ where $R_1(t)$ is the number density of the triplets of type $(+-+)$ per unit length, clearly exhibiting the asymmetry generated by the $(+-+)$ triplets. We also note that unlike the $\alpha=1$ case, the evolution equation (2) for the single-point correlation function involves two- and three-point correlations [via $R_1(t)$]. This hierarchy makes an exact solution difficult for $\alpha=0$.

It is useful at this point to summarize the main results obtained in this paper. Let us first highlight the contrast between the $\alpha=1$ (no kinetic disorder) and the $\alpha=0$ (with kinetic disorder) cases. For $\alpha=1$, due to the preserved symmetry between the up and down phases at all times, the average domain sizes of both "$+$" and "$-$" domains grow as $l_\pm(t)\sim t^{1/2}$ at late times [7]. Thus the average magnetization $m(t)=(l_+-l_-)/(l_++l_-)$ is a constant of motion [6]) and stays fixed at its initial value. In particular, if we start from an initial state where the spins are random (infinite temperature), the initial average magnetization is zero and stays zero at all subsequent times. In contrast, for the $\alpha=0$ case where the symmetry is dynamically broken, we find the following results.

(1) For $\alpha=0$, while domains of both phases continue to grow with time, they have different growth laws. The average domain sizes of the $+$ phases [denoted by $l_+(t)$] and $-$ phases [denoted by $l_-(t)$] coarsen at late times in the following manner: (i) $l_+(t)\approx\sqrt{\pi t}$ at late times and (ii) $l_-(t)\sim t^{1/2}\ln(bt)$ where b is a number depending on the initial volume fraction of the "$+$" spins which we will calculate explicitly (see below). In fact, the main result we show below is that the ratio of the two length scales behaves at late times as

$$\frac{l_-}{l_+}=\frac{1}{\epsilon}\frac{\ln(bt/t_0)}{\ln b}-1, \tag{3}$$

where ϵ is the initial volume fraction (need not be small) and t_0 is some initial time after which scaling starts holding. Equation (3) explicitly reflects the effect of broken symmetry. Thus due to the dynamically generated local bias, the "$-$" domains grow slightly faster than the "$+$" domains. We also point out that in contrast to the spin models studied in the context of glassy systems [2,8], the dynamics in our model does not freeze at zero temperature, rather the domains coarsen indefinitely in an infinite system.

(2) Consequently, the magnetization $m(t)=(l_+-l_-)/(l_++l_-)$ decays at late times as

$$m(t)=-1+\frac{2\epsilon\ln b}{\ln(bt/t_0)}. \tag{4}$$

(3) Evidently the number of "$+$" domains is the same as the number of "$-$" domains since they alternate on the 1D lattice. The density of domains of either "$+$" or "$-$" type per unit length, $N(t)=1/[l_-+l_+]$ decays as

$$\frac{N(t)}{N(t_0)}=\sqrt{\frac{t_0}{t}}\frac{\ln b}{\ln(bt)}. \tag{5}$$

We also compute explicitly two other densities that play a somewhat central role in our analysis.

(4) $r_1(t)$: Given that a "$-$" domain has occurred, the probability that it is of length 1. We show below that

$$r_1(t)=\frac{\sqrt{\pi}}{\sqrt{t}\ln(bt/t_0)}. \tag{6}$$

Note that the amplitude $\sqrt{\pi}$ is universal and is independent of volume fraction.

(5) $p_1(t)$: Given that a "$+$" domain has occurred, the probability that it is of length 1. We show that

$$p_1(t) = \frac{1}{2t} + \frac{1}{t \ln(bt/t_0)}. \tag{7}$$

Once again the amplitude 1 of the correction term is also universal and is independent of volume fraction.

The constant b in the above equations can also be computed exactly, and we show that

$$b = \exp\left(\frac{\sqrt{\pi}}{r_1(t_0)\sqrt{t_0}} \right). \tag{8}$$

III. CONNECTION TO GRANULAR COMPACTION AND REACTION DIFFUSION SYSTEMS

We now establish a one-to-one mapping between our spin model (with $\alpha = 0$) and a simple lattice model of granular compaction. Let us consider a $(1+1)$-dimensional granular packing where the grains are represented by square blocks. The pack consists of horizontal layers consisting of blocks and voids (see Fig. 1). We focus on the ''active'' layer, i.e., the first horizontal layer that is not fully compact with blocks as we go up from the bottom of the pile. Below this active layer, all layers are compact and remain compact under vertical tapping, i.e., their dynamics is completely frozen. All the activities take place in or above the ''active'' layer. We identify this ''active'' layer with the one-dimensional lattice of the spin model. This active layer consists of sequences of blocks (particles) and voids (holes). We identify a unit block or a particle in the active layer as a ''$-$'' spin of our spin model. Similarly a hole is identified as a ''$+$'' spin of the spin model.

As the system is tapped vertically, the particles in the active layer can undergo the following moves.

(1) In the interior of a row of consecutive particles in the active layer, there is no effect of tapping as the system is completely jammed there. The effect of vertical tapping is felt only at the edges of a particle cluster. The particles at the edges, under tapping, can move up or ''tap'' up from the active layer to the layers above the active layer (see Fig. 1). However, if the cluster consists of an isolated particle sandwiched between two holes, it has no place to move up, so it stays at its original location.

(2) Tapping also can ''roll'' off a particle residing in a layer above the active layer, into the active layer, at the edges of the supporting cluster (see Fig. 1).

Let us now see what the rates of spin flips in Eq. (1) in the spin model imply for the compaction model. The rate $W(+;++) = 0$ implies that if we have three consecutive holes in the active layer, then a particle cannot be deposited in the middle site under tapping. This is because a new particle can appear into the active layer only at the edges of the clusters of particles, but not in the middle of a cluster of voids. The rate $W(-;--) = 0$ implies that if we have three consecutive particles in the active layer, the middle particle cannot move up under tapping since it is completely jammed. The rate $W(+;-+) = 1/2$ signifies that if we have three consecutive sites in the active layer consisting, respectively, of a particle, hole, and a hole, then a particle can appear in the middle hole with rate 1/2. This is the ''rolling'' off move to the right at the edge of a cluster, as discussed above. Similarly $W(+;+-) = 1/2$ signifies the ''rolling'' off move to the left. The rate $W(-;+-) = 1/2$ implies that when we have three consecutive sites in the active layer consisting, respectively, of a hole, particle, and a particle, the middle particle can disappear (into the layers above) with rate 1/2. This is the ''tapping'' up move to the right from the edge of a cluster as discussed above. Similarly $W(-;-+) = 1/2$ signifies the rate of ''tapping'' up to the left from the edge of a cluster. The rate $W(+;--) = 1$ indicates that if we have a sequence of particle, hole, particle in the active layer, the middle hole can be filled up with a particle with rate 1. This is simply the addition of two ''rolling'' off rates from the left and the right of the middle hole. Finally the rate $W(-;++) = \alpha = 0$ implies that if we have a sequence of hole, particle, hole in the active layer, the middle particle cannot disappear. This is because an isolated particle has no place to move or ''tap'' up as discussed above. This last rate indeed breaks the particle-hole symmetry.

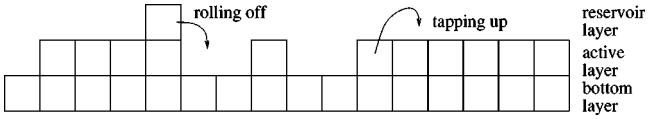

FIG. 1. Picture of rolling off (left domain) and tapping up (right domain) in the granular interpretation of the spin model; the solid squares represent particles.

Note that the rates $W(+;-+) = W(+;+-) = W(-;+-) = W(-;-+) = \frac{1}{2}$ correspond to the diffusion of domain walls in the spin model. In the compaction model, these are indeed the moves induced by the mechanical tapping. If these rates were zero, i.e., no tapping, then the dynamics in the active layer would freeze after all the isolated holes are filled up and the system will be stuck in a metastable configuration and hence the compaction will stop. These diffusion moves that lift the system out of a metastable configuration and the system continues to compactify, though extremely slowly. Identifying the ''$-$'' $(+)$ spins with particles (holes) in the granular model, it is easy to see that the average magnetization m in the spin model is related to the average particle density $\rho(t)$ in the granular model via the simple relation, $m(t) = 1 - 2\rho(t)$. Thus the result in Eq. (4) for the magnetization indicates that the density will grow to its fully compact value 1 as

$$\rho(t) = 1 - \frac{\epsilon \ln b}{\ln(bt/t_0)}, \tag{9}$$

at very late times, as observed in experiments on mechanically tapped granular media [3].

Other one-dimensional lattice models, notably the ''car parking'' model, have been previously used to explain the logarithmic relaxation in granular materials [9]. The local rules for the dynamics of particles in the car parking model are, however, quite different from those in our model. In the parking model, the rules for the particle motion are (i) a new particle can be absorbed only at sites containing a hole at a rate proportional to the number of neighboring particles and

(ii) a particle evaporates with a small rate (thus leaving behind a hole) if it has one and only one neighboring hole. While this model also exhibits an inverse logarithmic growth of the density, the dynamical moves (in particular, the fact that the desorption rate is infinitesimally small) are chosen in a somewhat *ad hoc* manner. In contrast, as explained via the mapping detailed above, our model incorporates the basic minimal microscopic moves of the particles that are observed in the compaction process.

In terms of the motion of the domain walls between "+" and "−" phases, our model can also be considered a model of two species reaction diffusion in one dimension. We note that in the case $\alpha=1$, the domain walls diffuse and annihilate upon contact. This corresponds to the process $A+A \rightarrow 0$ [7]. In the case $\alpha=0$, we need to distinguish between the two types of domain walls $-+\equiv A$ and $+-\equiv B$. Note that by definition (originating from a spin configuration) the A's and B's always occur alternately. Here both A's and B's diffuse as before; however when an A and a B meet, they annihilate only if A is to the left of B, otherwise there is hard core repulsion between them. We will show below that this hard core repulsion between the particles is a relevant interaction that changes the late time dynamics considerably. Recently the relevance of hard core repulsion between particles in reaction-diffusion systems has been explored in a number of contexts [10].

IV. SOME EXACT RESULTS

To start with we write down two exact relations that are derived directly from the Glauber dynamics (we shall see later that these exact relations are in fact respected by the IIA approximation).

Let $N(t)$ be the number of domains of either "+" or "−" spins per unit length. The density of kinks is therefore $2N(t)$. Let P_1 be the density of the triplets "−+−" per unit length, then clearly $p_1=P_1/N$. We note also that the only way in which a domain can be destroyed in an infinitesimal time step is by flipping an isolated "+" spin in a triplet "−+−." This gives

$$\frac{dN}{dt}=-P_1. \tag{10}$$

If $P(S_i)$ denotes the probability that the spin at site i takes the value S_i, the evolution of $P(S_i)$ depends only on the rates in Eq. (1) and the three-point probability distribution $P(S_{i-1},S_i,S_{i+1})$. The evolution of $P(S_i)$ is then given by

$$\frac{dP(S_i)}{dt}=\sum_{S_{i+1},S_{i-1}} W(-S_i;S_{i-1},S_{i+1})P(S_{i-1},-S_i,S_{i+1}) - \sum_{S_{i+1},S_{i-1}} W(S_i;S_{i-1},S_{i+1})P(S_{i-1},S_i,S_{i+1}). \tag{11}$$

Substituting $S_i=1$ in the above equation we find

$$\frac{dP(1)}{dt}=P(-1,-1,1)-P(-1,1,-1)-P(1,1,-1), \tag{12}$$

where we have used the evident left to right symmetries $P(-1,-1,1)=P(1,-1,-1)$ and $P(1,1,-1)=P(-1,1,1)$. We now observe that $P(-1,-1,1)+P(1,-1,1)=P(-1,1)$ and $P(-1,1,-1)+P(1,1,-1)=P(1,-1)=P(-1,1)$ and thus find

$$\frac{dP(1)}{dt}=-P(1,-1,1). \tag{13}$$

We note that $P(1,-1,1)$ is simply the probability that at a given site the spin is a "−" spin and its two neighbors are "+" spins, that is to say there is a "+−+" defect at the site considered. If we now sum this equation over each site on an interval of unit length on the lattice and recall that R_1 is the density of the "+−+" triplets per unit length, we obtain

$$\frac{dL_+}{dt}=-R_1, \tag{14}$$

where L_+ is the fraction of the "+" spins. Note that $L_+ + L_-=1$, where L_- is the fraction of the "−" spins. From Eq. (14) we obtain Eq. (2) for the evolution of the average magnetization m. Physically it is easy to see the origin of Eq. (14) as, on an average, the fraction of "+" spins can decrease only due to the blockage by "+−+" triplets.

Writing Eq. (14) in terms of $r_1=R_1/N$ and the average length of the "+" domains $l_+=L_+/N$, we obtain

$$\frac{\dot{N}}{N}=-\frac{\dot{l}_+ + r_1}{l_+}, \tag{15}$$

where $\dot{x}=dx/dt$. This equation can be integrated starting at some arbitrary time t_0 to give

$$\frac{N(t)}{N(t_0)}=\frac{l_+(t_0)}{l_+(t)}\exp\left(-\int_{t_0}^{t}\frac{r_1(t')}{l_+(t')}dt'\right). \tag{16}$$

Furthermore, if the volume fraction of the "+" phase is $L_+(t_0)=\epsilon$, then, using the relation $N(t)=1/[l_-(t)+l_+(t)]$ in Eq. (16), we find

$$\frac{l_-(t)}{l_+(t)}=\frac{1}{\epsilon}\exp\left(\int_{t_0}^{t}\frac{r_1(t')}{l_+(t')}dt'\right)-1, \tag{17}$$

clearly showing that the ratio $l_-(t)/l_+(t)$ is growing due to the presence of the triplets "+−+." Note that the asymmetry between the growth of "−" and "+" domains is evident due to the presence of the triplet defects "+−+" with density $R_1=r_1N$.

All the results presented above are exact. To derive the late time behavior of the model, we first consider below the IIA. We solve the IIA equations self-consistently and show that the IIA precisely predicts the results mentioned in Sec. II. Besides we shall argue that in the case where the initial

volume fraction of the "+" domains is small, i.e., $\epsilon \ll 1$, correlations do not develop between the domains if no correlations are present in the initial conditions and hence the IIA is exact to leading order in ϵ. At the end, we present a very simple heuristic argument which is also in agreement with these results.

V. IIA ANALYSIS

In this section we consider the IIA where correlations between neighboring domains are neglected. The IIA was used previously for the $\alpha=1$ case [11], yielding results in agreement, qualitatively as well as quantitatively to a fair degree of accuracy, with the exact results available [6,12]. Let $P_n(t)$ and $R_n(t)$ denote, respectively, the number density of the "+" and "−" domains of length n at time t. Note that $R_1(t)$ is the density of the triplet "+−+" as before. Let $N(t)=\Sigma_n P_n=\Sigma_n R_n$ denote the domain density of the "+" or "−" spins. Also the fractions $L_{\pm}(t)$ of "+" and "−" spins are given by $L_+(t)=\Sigma_n n P_n$ and $L_-(t)=\Sigma_n n R_n$ with $L_+(t)+L_-(t)=1$. During an infinitesimal time step Δt, $P_n(t)$ evolves as

$$P_n(t+\Delta t)=P_n(t)-\Delta t P_n(t)-\Delta t P_n(t)\left[1-\frac{R_1(t)}{N(t)}\right] + \Delta t P_{n+1}(t)+\Delta t P_{n-1}(t)\left[1-\frac{R_1(t)}{N(t)}\right]. \tag{18}$$

The right hand side of the above equation includes the various loss and gain terms. The second and the third terms describe, respectively, the loss due to the hopping inward and the hopping outward of the domain walls at the two ends of a "+" domain of size n. An outward hop can occur provided the neighboring domain in the direction of the hop is not an isolated "−" spin and this is ensured by the prefactor $(1-R_1/N)$ in the third term. The fourth and the last term describe similarly the corresponding gains. One can similarly write down the evolution equation for the $R_n(t)$'s. During an infinitesimal time step Δt, $R_n(t)$ evolves as

$$R_n(t+\Delta t)=R_n(t)-\Delta t R_n(t)-\Delta t R_n(t)\left[1-\frac{P_1(t)}{N(t)}\right] - 2\Delta t P_1(t)\frac{R_n(t)}{N(t)}+\Delta t R_{n+1}(t)+\Delta t R_{n-1}(t)\left[1 - \frac{P_1(t)}{N(t)}\right]+\frac{P_1(t)}{N^2(t)}\sum_{i=1}^{n-2} R_i(t)R_{n-i-1}(t); n\geq 2 \tag{19}$$

and

$$R_1(t+\Delta t)=R_1(t)-\Delta t R_1(t)\left[1-\frac{P_1(t)}{N(t)}\right]-2\Delta t P_1(t)\frac{R_1(t)}{N(t)} + \Delta t R_2(t). \tag{20}$$

The negative (loss) terms in Eq. (19) for domains of size $n \geq 2$ may be understood as follows. A domain of length n may be lost by the domain wall at either end jumping inwards with rate 1/2. This term is the second term in Eq. (19) and as there are two domain walls we have a factor of 2. A domain of length n may also be lost by a domain wall hopping outwards. This happens with rate 1/2 if the neighboring domain is not a triplet "−+−." The third term of Eq. (19) corresponds to this event, the factor $[1-P_1(t)/N(t)]$ is the probabilty of the absence of a triplet "−+−" as a neighboring domain. There is again a factor of 2 coming from the fact that there are two domain walls. However if a neighboring domain is of type "−+−" the outward jump towards this domain occurs with rate 1 (as the central + spin flips with rate 1). The term corresponding to these two events (from the right and left domain walls) is the fourth term in Eq. (19). The two first gain terms come from identical arguments and the last convolution term represents domain coalescence, where a domain of length n is formed with rate 1 from two "−" domains of length i and $n-i-1$ (where $1 \leq i \leq n-2$) if they are separated by a "−+−" triplet. Equation (20) is obtained in a similar fashion with the exception that the hard core repulsion generates a reflecting boundary condition. Taking the limit $\Delta t \to 0$ in the above equations, we obtain the IIA equations for the evolution of the domain densities

$$\frac{dP_n}{dt}=P_{n+1}+P_{n-1}-2P_n+\frac{R_1}{N}(P_n-P_{n-1}) \tag{21}$$

for all $n \geq 1$ with $P_0=0$ (absorbing boundary condition) and

$$\frac{dR_n}{dt}=R_{n+1}+R_{n-1}-2R_n-\frac{P_1}{N}(R_n+R_{n-1}) + \frac{P_1}{N^2}\sum_{i=1}^{n-2} R_i R_{n-i-1}; \quad n\geq 2$$

$$\frac{dR_1}{dt}=R_2-R_1-\frac{P_1}{N}R_1. \tag{22}$$

It is somewhat convenient to use the normalized variables $p_n=P_n/N$ and $r_n=R_n/N$. The average domain lengths are then given by $l_+(t)=\Sigma n p_n$ and $l_-(t)=\Sigma n r_n$ and the domain density $N(t)=1/[l_+(t)+l_-(t)]$. In terms of these normalized variables, the IIA equations are given by

$$\frac{dp_n}{dt}=p_{n+1}+p_{n-1}-2p_n+r_1(p_n-p_{n-1})+p_1 p_n \tag{23}$$

for all $n \geq 1$ with $p_0=0$ (absorbing boundary condition) and

$$\frac{dr_n}{dt}=r_{n+1}+r_{n-1}-2r_n-p_1 r_{n-1}+p_1\sum_{i=1}^{n-2} r_i r_{n-i-1}; \quad n\geq 2$$

$$\frac{dr_1}{dt}=r_2-r_1. \tag{24}$$

It is easy to check that the normalization condition $\Sigma p_n = \Sigma r_n = 1$ is satisfied by these two equations. We note that Eq. (23) or more clearly its unnormalized version in Eq. (21), i.e., $\dot{P}_n = P_{n+1} + P_{n-1} - 2P_n + r_1(P_n - P_{n-1})$, just represents the motion of a random walker on the positive side of a 1D lattice with a sink at the origin ($P_0 = 0$) and a time dependent drift term (proportional to r_1). To calculate $N(t)$ using Eq. (16), we need to evaluate two quantities from the IIA equations: (i) $r_1(t) = R_1/N$ and (ii) $l_+(t) = \Sigma n p_n$.

The two IIA equations above are coupled nonlinear equations with infinite number of variables and hence exact solution of Eqs. (23) and (24) are difficult. Our approach will be a combination of a scaling assumption and then rechecking this assumption for self-consistency. Consider first the r_n equation, i.e., Eq. (24). On the right hand side, we will first ignore the diffusion term, solve for the rest, and show that indeed neglecting the diffusion term was justified in the first place. This is self-consistency. Ignoring the diffusion term, we have the following equation:

$$\frac{dr_n}{dt} = p_1 \left[\sum_{i=1}^{n-2} r_i r_{n-i-1} - r_{n-1} \right], \tag{25}$$

with the reflecting boundary condition, $r_2 = r_1$. It is now easy to see that Eq. (25) admits a scaling solution, $r_n(t) \approx \lambda(t) \exp[-n\lambda(t)]$, where $\dot{\lambda}(t) = -p_1(t)\lambda(t)$. Using the exact relation $dN/dt = -p_1 N$, we get $\lambda(t) = \lambda(t_0) N(t)/N(t_0)$. Note that we still do not know what $N(t)$ is. Now let us substitute this solution to estimate the diffusion term that had been neglected in the first place. Clearly, the diffusion term $T_{diff} = r_{n+1} + r_{n-1} - 2r_n \sim O(\lambda^3(t))$, whereas the other terms [for example, the left hand side of Eq. (24)] typically scale as $\sim O(\dot{\lambda}(t)) \sim O(p_1(t)\lambda(t))$. Thus, in order to be self-consistent in neglecting the diffusion term, we need to have $p_1(t) \gg \lambda^2(t) \sim N^2(t)$. We will see that this condition is actually satisfied once we derive the expression for $N(t)$. This just means that the diffusion terms only contribute to the corrections to the leading scaling behavior.

From the above analysis, we find to leading order for large t, $r_1 \approx \lambda(t) = \lambda(t_0) N(t)/N(t_0)$, i.e., $r_1(t) \approx r_1(t_0) N(t)/N(t_0)$. We now need to evaluate the other remaining quantity, $l_+(t) = \Sigma n p_n$. For this we now turn to the p_n equation, Eq. (23). In this equation, we will again first ignore the drift term $r_1(p_n - p_{n-1})$, solve for the rest, and check that indeed the neglect of the drift term was justified. Ignoring the drift term, we get

$$\frac{dp_n}{dt} = p_{n+1} + p_{n-1} - 2p_n + p_1 p_n \tag{26}$$

with the absorbing boundary condition $p_0 = 0$. This equation can be solved exactly. Indeed it also admits a scaling solution $p_n(t) = t^{-1/2} f(nt^{-1/2})$, where the scaling function (normalized to unity) is given by $f(x) = (x/2)\exp(-x^2/4)$. Now let us estimate the drift term that was neglected. Clearly the drift term $r_1(p_n - p_{n-1}) \sim O(r_1/t) \sim O(N(t)/t)$ since $r_1 \sim N(t)$ from above. The other terms in the Eq. (23) [for example, the left hand side of Eq. (23)] is of order $t^{-3/2}$ at late times. Thus, for self-consistency in neglecting the drift term, we need to show that $t^{-3/2} \gg N(t)/t$. We will again see that this condition is indeed also satisfied once we obtain the expression for $N(t)$. From this form of $p_n(t)$, we thus obtain, to leading order for large t, $l_+(t) = \Sigma n p_n \approx t^{1/2} \int_0^\infty x f(x) dx$. Using $f(x) = (x/2)\exp(-x^2/4)$, and doing the integral we finally find $l_+(t) \approx \sqrt{\pi t}$ for large t.

Using these two results (i) $r_1(t) = r_1(t_0) N(t)/N(t_0)$ and (ii) $l_+(t) = \sqrt{\pi t}$ in the exact equation, Eq. (16), and differentiating with respect to t, we find a differential equation for $N(t)$,

$$\frac{d}{dt}[\sqrt{t} N(t)] = -\frac{r_1(t_0)}{\sqrt{\pi} N(t_0)} N^2(t). \tag{27}$$

Introducing the dimensionless variable

$$S(t) = \sqrt{\frac{t_0}{t}} \frac{N(t)}{N(t_0)} \tag{28}$$

in Eq. (27) we find

$$\frac{dS}{dt} = -\ln(b) \frac{S^2}{t}, \tag{29}$$

where $\ln(b) = \sqrt{\pi}/(r_1(t_0)\sqrt{t_0})$. Integrating Eq. (29) we find

$$S(t) = \frac{\ln(b)}{\ln(bt/t_0)}, \tag{30}$$

we thus obtain the result

$$\frac{N(t)}{N(t_0)} = \sqrt{\frac{t_0}{t}} \frac{\ln(b)}{\ln(bt/t_0)}. \tag{31}$$

Substituting this result in the expression $r_1(t) \approx r_1(t_0) N(t)/N(t_0)$, we get

$$r_1(t) = \frac{\sqrt{\pi}}{\sqrt{t}\ln(bt/t_0)}. \tag{32}$$

Next we use the late time result, Eq. (31), in the exact relation, Eq. (10), and find

$$p_1 = \frac{1}{2t} + \frac{1}{t\ln(bt/t_0)}. \tag{33}$$

Let us check the two self-consistency conditions, (a) $p_1(t) \gg \lambda^2(t) \sim N^2(t)$ and (b) $t^{-3/2} \gg N(t)/t$. Using the expression for p_1 from Eq. (33) and that of $N(t)$ from Eq. (31), it is immediately evident that indeed these two conditions are satisfied for large t. Thus our whole approach has been completely self-consistent and the IIA results are precisely those mentioned in the Introduction. Note also that t_0 must be sufficiently large such that both scaling laws $r_1(t) \sim N(t)$ and $l_+(t) \sim \sqrt{\pi t}$ start holding for $t > t_0$.

VI. ZERO VOLUME FRACTION LIMIT

In this section, we show that the IIA results essentially become exact in the zero volume fraction limit of the "+" phase, i.e., in the limit $\epsilon \to 0$. Suppose we start from an initial condition such that $r_n(0) = \epsilon(1-\epsilon)^n$ and $p_n(0) = \delta_{n,1}$. This means that in the initial condition, the average length of the "−" domains, $l_-(0) \sim 1/\epsilon$, whereas $l_+(0) = 1$. Thus the "−" domains are typically much larger than the "+" domains, in the limit $\epsilon \to 0$. Also initially all the domains are completely uncorrelated. So the picture is as follows. We have little droplets of the "+" phase in a sea of the "−" phase. Besides, one can also compute the initial density of domains of either "+" or "−" types. It is given by $N(0) = 1/\epsilon(1-\epsilon) \sim 1/\epsilon$ to leading order in ϵ.

Now let us consider the time evolution of the system starting from this initial condition. As time increases, the "+" domains will certainly grow in size. But a typical "+" domain will disappear (via the absorbing boundary condition) much before encountering other "+" domains, i.e., before feeling the presence of the constraint due to triplets "+ − +." The probability of such an event is $O(\epsilon)$. Thus effectively, the dynamics of the system will proceed via eating up of the "+" domains. Hence, if there is no correlation between domains in the initial condition, the dynamics is not going to generate correlations between them. This is precisely what happens in the zero-temperature dynamics of the q state Potts model in 1D in the limit $q \to 1^+$ [13,11].

Thus in this limit, the evolution of the "−" domains is governed by the exact equation

$$\frac{dr_n}{dt} = p_1 \left[\sum_{i=1}^{n-2} r_i r_{n-i-1} - r_{n-1} \right], \tag{34}$$

which is same as the IIA equation, Eq. (24), without the diffusion term. Starting from the initial condition $r_n(0) = \epsilon(1-\epsilon)^n$, one can solve the above equation for any t exactly to leading order in ϵ. It turns out that, to leading order in ϵ, Eq. (34) admits a solution $r_n(t) = \mu(t)[1-\mu(t)]^{n-1}$, where $\mu(t) = \epsilon \exp[-\int_0^t p_1(t')dt']$. Using once again the exact equation $dN/dt = -p_1 N$, we find $\mu(t) = \epsilon N(t)/N(0) + O(\epsilon^2) = N(t) + O(\epsilon^2)$ where we have used $N(0) = 1/\epsilon + O(\epsilon^2)$. Thus we get

$$r_1(t) = \mu(t) = N(t) + O(\epsilon^2). \tag{35}$$

Now, let us consider the evolution of the "+" domains. Since a typical "+" domain never encounters (to leading order in ϵ) any other "+" domain and hence does not feel the constraint due to r_1's, the effective dynamics of a "+" domain is that of a single "+" domain immersed in a sea of the "−" phase. Let P_n denote the probability that such a domain is of length n. Then, to leading order in ϵ, P_n's clearly evolve by the simple diffusion equation

$$\frac{dP_n}{dt} = P_{n+1} + P_{n-1} - 2P_n, \tag{36}$$

with the absorbing boundary condition $P_0 = 0$. The normalized conditional probability $p_n = P_n/N$ with $N = \Sigma P_n$ then satisfies the equation

$$\frac{dp_n}{dt} = p_{n+1} + p_{n-1} - 2p_n + p_1 p_n, \tag{37}$$

same as Eq. (26). This equation has to be solved with the initial condition $p_n(0) = 1$. It can be solved exactly. Without writing the explicit solution, we just mention the result for $\Sigma n p_n$. We find that for large t and leading order in ϵ,

$$l_+(t) = \sum n p_n \approx \sqrt{\pi t} + O(\epsilon). \tag{38}$$

Using the results from Eqs. (35) and (38), i.e., (i) $r_1(t) = N(t) + O(\epsilon^2)$ and (ii) $l_+(t) \approx \sqrt{\pi t} + O(\epsilon)$ in the exact equation, Eq. (16), we once again recover all the IIA results of the preceding section, with $b = \sqrt{\pi}/\epsilon$.

Hence IIA becomes exact in the $\epsilon \to 0$ limit. This is not surprising as the dynamics in this limit does not generate correlations if there are none in the initial condition.

VII. OTHER VOLUME FRACTIONS

For finite initial volume fraction of the "+" phase, the IIA cannot be exact since the diffusion of kinks correlates the domains as time progresses, even if the domains had no correlations to start with. However, the volume fraction of the "+" phase decreases monotonically with time according to the exact equation, Eq. (14). Thus at very late times when the volume fraction $L_+(t)$ is very small, the effective fixed point picture of the system is very similar to the $\epsilon \to 0$ limit picture, i.e., small "+" domains immersed in the sea of "−" domains. The only difference is that the big "−" domains may now be correlated. However, it is very likely (though we cannot prove this rigorously) that the correlations between domains are very small at very late times and therefore the IIA results summarized in the points (1) to (5) of the Introduction become asymptotically exact. The numerical results reported in Ref. [5] appear to confirm this fact.

Actually if this late time fixed point picture is correct, then one can derive all the results from a very simple heuristic argument, as is presented below.

VIII. HEURISTIC APPROACH

The heuristic picture is as follows. Due to the facts that "+" domains can grow only by diffusion and the "−" domains grow by diffusion and coalescence, one expects that at late times the ratio $l_+(t)/l_-(t) \to 0$; this is also clear from Eq. (14). Therefore at late times one expects to find "+" domains sandwiched between much larger "−" domains. Consequently the "+" domains, to a first approximation, never encounter the "+ − +" triplets at late times and hence diffuse freely and annihilate via the disappearance of the "− + −" triplets. Therefore the length of a "+" domain behaves as a diffusion with a killing boundary condition at the origin. Solving the discrete diffusion equation corre-

sponding to this picture [see Eq. (37)], one finds that at late times

$$l_+(t) \approx \sqrt{\pi t}. \tag{39}$$

If one now views the system at the length scale of the "$-$" domains $l_-(t) \gg l_+(t)$, one sees long stretches of "$-$" domains occasionally interrupted by "$+$" domains that are now shrunk to a single point when viewed from the length scale of the "$-$" domains. The rate of occurrence of these points per unit length $\lambda(t)$ is clearly proportional to $N(t)$, i.e., the kink density. Note that r_n is simply the conditional probability: Given that a "$-$" domain has occurred, what is the probability that it is of length n. Now if one assumes that these punctual "$+$" domains (or the points) are distributed randomly, one finds that r_n is simply given by the geometric distribution $r_n = \lambda(t)[1-\lambda(t)]^{n-1}$, where $\lambda(t) = cN(t)$ for some constant c. We therefore find

$$r_1(t) \approx cN(t), \tag{40}$$

for late times. Furthermore, if we denote by t_0 a large time after which this picture becomes valid, we may write

$$r_1(t) \approx r_1(t_0)\frac{N(t)}{N(t_0)}. \tag{41}$$

These two expressions for r_1 and $l_+(t)$ once again are same as obtained by a more careful analysis of the IIA equations and when substituted in Eq. (16), they give the same IIA results once again. Thus the basic assumption of this heuristic picture is that the "$+$" domains occur randomly, which seems like an accurate description at late times.

IX. CONCLUSIONS

In this paper, we have presented detailed analytical studies on a simple one-dimensional kinetically constrained Ising model that was introduced in Ref. [5]. The kinetic constraints in this model are local and dynamically generated. The effect of these constraints was shown to slow down the dynamics rather dramatically. We have shown that the average magnetization in this model decays extremely slowly with time in an inverse logarithmic fashion to its final saturation value. This kind of inverse logarithmic law was observed in the behavior of the density of granular material in experiments on granular compaction [3] and was also saw in numerical simulations of various lattice based and "tetris-" like models [14]. There have been some theoretical arguments proposing various mechanisms responsible for this slow compaction [15,9]. These include the free volume argument [15] and the argument based on an analogy with car parking models with the somewhat *ad hoc* assumption that the cars "depark" from a lane at an infinitesimal rate [9]. In contrast, in this paper we have mapped our kinetic Ising model *directly* to a lattice model of granular compaction that incorporates the basic minimal microscopic moves in the compaction process. The average magnetization $m(t)$ in the Ising model, via this mapping, gets related to the density of compaction $\rho(t)$ in the granular model as $\rho(t) = (1-m)/2$. Hence, besides having nontrivial behavior and yet analytically solvable, our toy model of granular compaction correctly reproduces the inverse logarithmic time dependence seen in the experiments [3] and thereby proposes an entirely different mechanism for this slow compaction, quite different from the previous models such as the car parking models. It is also interesting to note a study of compaction in the tetris model [16] shows that at late times the activity of the system, leading to compaction, occurs at boundaries between domains that can be identified in the system. The image of compaction as a kinetically hindered coarsening process thus appears to be quite robust.

From a somewhat broader perspective, our work addresses a general question: what is the effect of kinetically generated disorders on the coarsening dynamics in domain growth problems? In the present work, we have studied a specific type of kinetic disorder, namely, a dynamically generated local magnetic field. This field acts locally on the topological defects responsible for the coarsening process (in this case, simple domain walls). Our study suggests that such kinetic disorders, while slowing down the dynamics drastically, do not altogether inhibit the coarsening process as found in other constrained kinetic Ising models [2]. The domain growth problems are rather common and occur in various physical systems [4]. Our work, therefore, opens up the possibility of studying the slowing down in coarsening dynamics due to kinetic disorders in many of these systems. For example, it would be interesting to study the effect of dynamically generated local fields in higher-dimensional Ising models, in $O(n)$ vector spin models and in liquid crystals, to mention a few. It is possible to have other types of local kinetic disorders than the one studied here and it would also be interesting to study their effect in coarsening systems.

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[1] J.P. Bouchaud, L.F. Cugliandolo, J. Kurchan, and M. Mézard, in *Spin Glasses and Random Field*, edited by A.P. Young (World Scientific, Singapore, 1998).

[2] G.H. Fredrickson, Annu. Rev. Phys. Chem. **39**, 149 (1988); J. Jackle and S. Eisinger, Z. Phys. B: Condens. Matter **84**, 115 (1991); J.D. Shore, M. Holzer, and J.P. Sethna, Phys. Rev. B **46**, 11 376 (1992); J. Kisker, H. Reiger, and H. Schreckenberg, J. Phys. A **27**, L853 (1994); F. Ritort, Phys. Rev. Lett. **75**, 1190 (1995); M.E.J. Newman and C. Moore, Phys. Rev. E **60**, 5068 (1999).

[3] E.R. Nowak, J.B. Knight, E. Ben-Naim, H.M. Jaeger, and S.R. Nagel, Phys. Rev. E **57**, 1971 (1998).

[4] A.J. Bray, Adv. Phys. **43**, 357 (1994).
[5] S.N. Majumdar, D.S. Dean, and P. Grassberger, Phys. Rev. Lett. **86**, 2301 (2001).
[6] R.J. Glauber, J. Math. Phys. **4**, 294 (1963).
[7] A.J. Bray, J. Phys. A **22**, L67 (1990); J.G. Amar and F. Family, Phys. Rev. A **41**, 3258 (1990); S.N. Majumdar and D.A. Huse, Phys. Rev. E **52**, 270 (1995).
[8] P. Sollich and M.R. Evans, Phys. Rev. Lett. **83**, 3238 (1999).
[9] P.L. Krapivsky and E. Ben-Naim, J. Chem. Phys. **100**, 6778 (1994); X. Jin, G. Tarjus, and J. Talbot, J. Phys. A **27**, L195 (1994); E. Ben-Naim, J.B. Knight, and E.R. Nowak, Physica D **123**, 380 (1998).
[10] S. Kwon, J. Lee, and H. Park, Phys. Rev. Lett. **85**, 1682 (2000); G. Odor and N. Menyhard, Phys. Rev. E **61**, 6404 (2000); F. van Wijland, *ibid.* **63**, 022101 (2001).
[11] P.L. Krapivsky and E. Ben-Naim, Phys. Rev. E **56**, 3788 (1997).
[12] B. Derrida and R. Zeitak, Phys. Rev. E **54**, 2513 (1996).
[13] C. Sire and S.N. Majumdar, Phys. Rev. E **52**, 244 (1995).
[14] E. Caglioti, V. Loreto, H.J. Herrmann, and M. Nicodemi, Phys. Rev. Lett. **79**, 1575 (1997); M. Nicodemi, J. Phys. I **17**, 1535 (1997); M. Piccioni, M. Nicodemi, and S. Galam, Phys. Rev. E **59**, 3858 (1999).
[15] J.B. Knight *et al.*, Phys. Rev. E **51**, 3957 (1995); T. Boutreux and P.G. de Gennes, Physica A **244**, 1 (1997).
[16] A. Baldassarri, S. Krishnamurthy, V. Loreto, and S. Roux, Phys. Rev. Lett. **87**, 224301 (2001).