PROBLEM 9

Anderson on Bethe Lattice (2/2)

$\boxed{1}$ IMAGINARY APPROXIMATION & DISTRIBUTIONAL EQUATIONS

Under all the approximation mentioned in the text,

$$\Gamma_a = t^2 \sum_{b\in\partial a} \frac{\Gamma_b+\eta}{W^2V_b^2 + (\Gamma_b+\eta)^2} \approx \frac{t^2}{W^2} \sum_{b\in\partial a} \frac{\Gamma_b+\eta}{V_b^2} \qquad (*)$$

because we assume $\Gamma_b \simeq \eta \ll 1$.

Since, as remarked above, the Γ_b are all independent and identically distributed with density $P_\Gamma(\Gamma)$, the identity $(*)$ becomes a self consistent equation for $P_\Gamma(\Gamma)$.

In particular,

$$P_\Gamma(\Gamma) = \overline{\delta\left(\Gamma - \left(\frac{t}{W}\right)^2 \sum_{b\in\partial a} \frac{\Gamma_b+\eta}{V_b^2}\right)}$$

which explicitly reads:

$$P_\Gamma(\Gamma) = \int \prod_{b=1}^{K} dV_b\, p(V_b) \int \prod_{b=1}^{K} d\Gamma_b\, P_\Gamma(\Gamma_b)\, \delta\left(\Gamma - \frac{t^2}{W^2} \sum_{b=1}^{K} \frac{\Gamma_b+\eta}{V_b^2}\right)$$

The Laplace transform is:

$$\Phi(s) = \int_0^\infty d\Gamma \, P_\Gamma(\Gamma) e^{-s\Gamma} = \int \prod_{b=1}^{K} d\Gamma_b \, P_\Gamma(\Gamma_b) \cdot \int \prod_{b=1}^{K} dV_b \, p(V_b) \, e^{-s\left(\frac{t}{W}\right)^2 \sum_{b=1}^{K} \frac{\Gamma_b + \eta}{V_b^2}}$$

$\leftarrow$ independence

$$= \left[\int d\Gamma \, P_\Gamma(\Gamma) \int dV \, p(V) \, e^{-s\frac{t^2}{W^2} \frac{(\Gamma+\eta)}{V^2}} \right]^K$$

$$= \left[\int dV \, p(V) \, e^{-s\left(\frac{t}{W}\right)^2 \frac{\eta}{V^2}} \, \Phi\left(\frac{s t^2}{W^2 V^2}\right) \right]^K$$

2 THE STABILITY ANALYSIS

- Assume $P_\Gamma(\Gamma) \sim \Gamma^{-\alpha}$ when $\Gamma \gg 1$.

First, it holds in full generality:

$$\lim_{s\to 0} \Phi(s) = \lim_{s\to 0} \int_0^\infty d\Gamma \, e^{-s\Gamma} P_\Gamma(\Gamma) = \int_0 d\Gamma \, P_\Gamma(\Gamma) = 1$$

by normalization.

Consider $\left(\Phi(s) - 1\right) = \int_0^\infty d\Gamma \, P_\Gamma(\Gamma) \left(e^{-s\Gamma} - 1\right)$

Assume r has some dimension $[r]$.
Because the exponent has to be adimensional,
$[s] = [r]^{-1}$.

Now, for $|s| \ll 1$ the integral is mostly contributed by $r \gg 1$, when $P_r(r) \sim r^{-\alpha}$. One has:

$$\left[\Phi(s) - 1\right] = \left[dr \; P_r(r)\right] = [r]^{1-\alpha} = [s]^{\alpha-1}$$

Thus, $\Phi(s) = 1 - A|s|^{\beta}$ $\quad$ $\beta = \alpha - 1$, $A > 0$

(the sign is because $\Phi(s) \leq 1$)

<u>Extra</u>: why one expects r to have power law tails, $P_r(r) \sim r^{-1-\alpha}$?

In first approximation, $r \sim 1/V^2$ and

$$P_r(r) \sim \int dV \, P(V) \, \delta\left(r - 1/V^2\right) \sim \frac{P(V)}{2} |V|^3 \Big|_{V = r^{-1/2}}$$

$$\sim \frac{1}{r^{3/2}} P\left(r^{-1/2}\right) \overset{r \gg 1}{\sim} 1/r^{3/2}$$

which suggests $\alpha = 1/2$

- The equation for $\Phi(s)$ is:

$$\Phi(s) = \left[\int dV\, p(V)\, e^{-\frac{s\tau^2\eta}{V^2}}\, \Phi\left(\frac{s\tau^2}{V^2}\right) \right]^K$$

which implies: $(s>0,\ \tau = t/W)$

$$1 - A s^\beta = \left[\int dV\, p(V)\, e^{-\frac{s\tau^2\eta}{V^2}} \left(1 - A\left|\frac{\tau}{V}\right|^{2\beta} s^\beta\right) \right]^K$$

$$= \left[1 + O(s) - A s^\beta \int dV\, \underset{\text{small}}{p(V)}\, e^{-s\tau^2\eta/V^2} \left|\frac{\tau}{V}\right|^{2\beta} \right]^K$$

$$\simeq 1 - AK s^\beta \int dV\, p(V) \left|\frac{\tau}{V}\right|^{2\beta} + o(s^\beta)$$

The two sides match provided that $\exists\,\beta$:

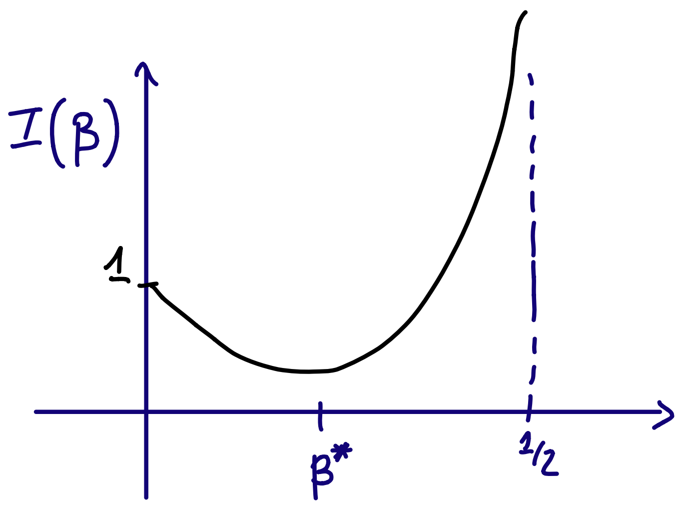
$$\int dV\, p(V) \left(\frac{\tau}{|V|}\right)^{2\beta} = 1/K.$$

3 CRITICAL DISORDER

For the uniform distribution:

$$I(\beta) = \int_{-1/2}^{1/2} dV \left|\frac{t}{V}\right|^{2\beta} = \left(\frac{t}{W}\right)^{2\beta} \int_{-1/2}^{1/2} dV \, 1/|V|^{2\beta}$$

$$= \left(\frac{2t}{W}\right)^{2\beta} \frac{1}{1-2\beta} = \frac{e^{2\beta \log(2t/W)}}{1-2\beta}$$



$$I(\beta) \xrightarrow{\beta=0} 1$$

$$I(\beta) \xrightarrow{\beta=1/2} \infty$$

$$I'(\beta) = 2\log\left(\frac{2t}{W}\right) I(\beta) + \frac{2 \cdot I(\beta)}{1-2\beta} = 2I(\beta)\left[\log\left(\frac{2t}{W}\right) + \frac{1}{1-2\beta}\right]$$

This vanishes when $\beta = \beta^* = \left(\frac{1}{\log\left(\frac{2t}{W}\right)} + 1\right)\frac{1}{2}$

And:

$$I(\beta^*) = \left[e^{\log\left(\frac{2t}{W}\right)+1}\right]\left(-\log\left(\frac{2t}{W}\right)\right) = \frac{2te}{W}\log\left(\frac{W}{2t}\right)$$

when W decreases, $I(\beta^*)$ increases: eventually, for W small enough it will reach $1/K$ (where $K \geq 2$), & localization becomes unstable

In particular, this happens when:

$$2e\left(\frac{t}{W}\right)_c \log\left(\frac{1}{2}\left(\frac{W}{t}\right)_c\right) = \frac{1}{K} \implies W_c = K \cdot 2te \log\left(\frac{W_c}{2t}\right)$$

Iterating and assuming $K \gg 1$

$$\left(\frac{W_c}{t}\right) = K\, 2\, e \log(K).$$

The critical disorder increases with the connectivity K of the graph: when K is larger, there are more "directions" along which the particle can move, and one needs a stronger disorder to localize it. Increasing K is like increasing dimensionality: localization becomes more difficult, i.e. it requires a stronger disorder.