Exercises 9-10: Glasses and Replicas

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Exercise 9. The annealed free energy of the spherical p-spin

Consider the spherical p-spin model discussed in class.

1. At variance with the REM, in the spherical p-spin model the energies at different configurations are correlated. Show that

$$\overline{E(\vec{\sigma})E(\vec{\tau})} = N\, q(\vec{\sigma},\vec{\tau})^p + o(N), \qquad \text{where} \qquad q(\vec{\sigma},\vec{\tau}) = \frac{1}{N}\sum_{i=1}^{N} \sigma_i \tau_i \tag{1}$$

   is the overlap between the two configurations. Why can we say that for $p \to \infty$ this model converges to the REM?
   Hint: Use that, when N is large, $p! \sum_{i_1<i_2<\cdots<i_p} X_{i_1\cdots i_p} = \sum_{i_1\neq i_2\neq\cdots\neq i_p} X_{i_1\cdots i_p} \approx \sum_{i_1,i_2,\cdots i_p} X_{i_1\cdots i_p}$ if $X_{i_1\cdots i_p}$ is a symmetric function of the indices.

2. Show that computing $\overline{Z}$ boils down to computing the average $\overline{e^{-\beta J_{i_1\cdots i_p}\sigma_{i_1}\cdots\sigma_{i_p}}}$, which is a Gaussian integral. Compute this average.
   Hint: if X is a centered Gaussian variable with variance v, then $\overline{e^{\alpha X}} = e^{\frac{\alpha^2 v}{2}}$.

3. The sphere $\mathcal{S}_N$ of radius $\sqrt{N}$ in dimension N has volume $2N^{\frac{N}{2}}\pi^{\frac{N}{2}}/\sqrt{N}\,\left(\frac{N}{2}\right)!$. Use the large-$N$ asymptotic of this to conclude the calculation of the annealed free energy:

$$f_{\rm a} = -\frac{1}{\beta}\left(\frac{\beta^2}{2} + \frac{1}{2}\log(2\pi e)\right).$$

   This result is only slightly different with respect to the annealed free-energy of the REM: can you identify the source of this difference?
   Hint: Use Stirling's formula.

4. In class we computed the quenched free-energy of the spherical p-spin, using the replica trick and thus computing $\overline{Z^n}$: how is the annealed free-energy obtained through that calculation?

Exercise 10. Susceptibilities, magnetic response, and glassiness

The magnetic susceptibility, that measures the response of the system to changes of the magnetic field, is defined as

$$\chi_{eq} = \lim_{h\to 0} \frac{dm(h)}{dh}$$

where $m(h) = \lim_{N\to\infty} m_N(h)$ is the magnetization at a given inverse temperature β and external field h, and $m_N(h)$ its finite-size counterpart.

1. Consider the mean field Ising model. Using the self-consistent equation for the magnetization $m = \tanh[\beta(h+m)]$, show that χ_{eq} diverges at the transition temperature $\beta = \beta_c = 1$ to the ferromagnetic phase.

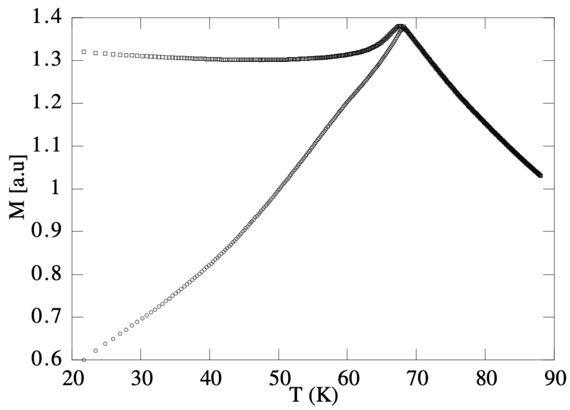

Figure 1: Figure taken from C. Djurberg, K. Jonason, P. Nordblad, *Magnetic Relaxation Phenomena in a CuMn Spin Glass*, https://arxiv.org/abs/cond-mat/9810314

2. The Fluctuation-Dissipation Theorem (FDT) establishes that the response of the system to small perturbations and its correlations at equilibrium are related by

$$\chi_N = \frac{dm_N(h)}{dh}\Big|_{h=0} = \frac{\beta}{N}\sum_{ij}\overline{\langle\sigma_i\sigma_j\rangle_c} = \frac{\beta}{N}\sum_{ij}\overline{\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle}. \tag{2}$$

Consider a mean-field system with a spin glass phase for $T < T_c$. Justify why it holds $\overline{\langle\sigma_i\sigma_j\rangle_c} = 0$ for $i \neq j$. Using the decomposition of the Boltzmann measure into pure states, show that

$$\chi_{eq} = \lim_{N\to\infty}\chi_N = \beta\left(1 - \int dq\,\overline{P_\beta(q)}\,q\right), \tag{3}$$

where $\overline{P_\beta(q)}$ is the average overlap distribution discussed in class.

3. What is $\chi_{\rm eq}(T)$ in the high- and low-temperature phases of the REM? Give the explicit expression and make a sketchy plot. Is the behavior consistent with what discussed in class about the REM? Does it diverge at the transition?

4. The quantity χ_{eq} is the response that one would measure if the system is prepared at equilibrium, then a small magnetic field is applied and the system is given enough time to reach the new equilibrium state. Justify why, if the system is not given enough time to re-equilibrate before measuring the response, the response that one measures is

$$\chi_{LR}(T) = \frac{1}{T}\left(1 - q_{\rm EA}(T)\right).$$

For the spherical p-spin model, do you expect this response to be higher or lower than $\chi_{\rm eq}(T)$ defined in (3)? Can you motivate why, physically?

5. Figure 1 shows experimental measurements of the magnetic susceptibility in a spin-glass. The two curves correspond to two different protocols: (a) ZFC (zero-field cooled) protocol: cool the system to low T, add a very small magnetic field when the system is already at the final low temperature; (b) FC (field-cooled): cool the system in presence of a small magnetic field and compare the observed magnetization with the one measured without this small magnetic field. Which of the the two susceptibilities χ_{eq}, χ_{LR} describe each of these two experimental protocols? Why?