Exercises 11-12: Dynamics

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Exercise 11: Force-Controlled Fully Connected Model

We consider the fully connected depinning automaton in the thermodynamic limit $N \to \infty$ under constant force F. All thresholds are set to $f_Y = 1$. Jump lengths Δ are independent random variables drawn from a distribution $g(\Delta)$.

The distance to instability is

$$x_i = 1 - (h_{\text{CM}} - h_i) - F.$$

In the thermodynamic limit the system is fully characterized by the distribution $P(x)$.

1. **Distribution of pinned distances**

   Assume the system is in a stationary state. A pinned block i is located at h_i^{stable}, the first trap to the right of h_{int}, where $x(h_{\text{int}}) = 0$.

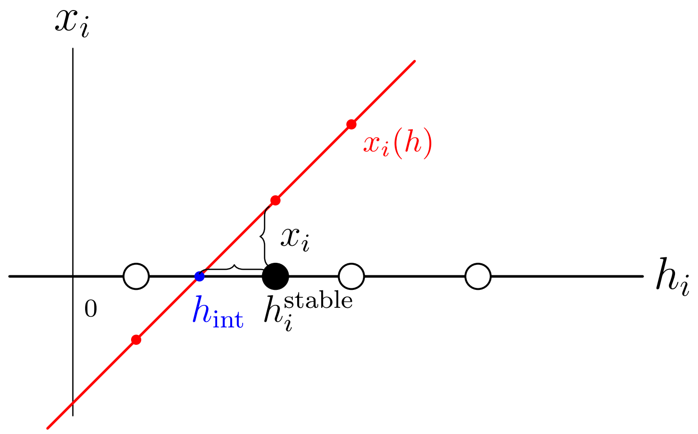

Figure 1: The intersection point h_{int} separates unstable ($x < 0$) and stable ($x > 0$) regions. A pinned block sits at the first trap to the right of h_{int}.

   Its distance to instability is therefore

$$x_i = h_i^{\text{stable}} - h_{\text{int}}.$$

   Why can h_{int} be chosen uniformly?

   In the stationary regime the interface has explored the disorder landscape and all trap positions are statistically equivalent along the h axis. Therefore h_{int} is uniformly distributed (translational invariance of disorder).

   Length-biased sampling.

   Consider a trap sequence with spacings Δ drawn from $g(\Delta)$.

- The probability that a uniformly chosen point falls inside an interval of length Δ is proportional to its length:
$$\mathrm{Prob}(\Delta) \propto \Delta\, g(\Delta).$$
- Normalizing gives
$$\frac{\Delta g(\Delta)}{\bar{\Delta}}, \qquad \bar{\Delta} = \int_0^\infty \Delta g(\Delta) d\Delta.$$
- Conditioned on Δ, the offset to the next trap is uniform on $[0, \Delta]$.

Show that
$$P(x) = \frac{1}{\bar{\Delta}} \int_x^\infty g(\Delta) d\Delta.$$

2. **Mean distance and critical force**

Compute
$$\bar{x} = \int_0^\infty x P(x) dx$$
and show that
$$\bar{x} = \frac{\overline{\Delta^2}}{2\bar{\Delta}}.$$
Using the definition of x_i, deduce that a pinned stationary state exists only if
$$F \leq F_c, \qquad F_c = 1 - \frac{\overline{\Delta^2}}{2\bar{\Delta}}.$$

3. **Evolution equation**

Using the update rule, show that
$$P_{t+1}(x) = P_t(x + v^{t+1}) H(x + v^{t+1}) + \int_0^\infty d\Delta\, P_t(x + v^{t+1} - \Delta) g(\Delta) H(-x - v^{t+1} + \Delta).$$

4. **Velocity–force relation**

Assume a stationary state with velocity v.

First moment.

Multiply the stationary equation by x and integrate to show
$$v = \bar{\Delta} \int_{-\infty}^0 P(x) dx.$$
Interpret this relation physically.

Second moment.

Multiply the stationary equation by x^2 and integrate.

Hints:

- Use
$$\bar{x} = 1 - F.$$
- Split averages:
$$\langle x\Theta(x) \rangle, \qquad \langle x\Theta(-x) \rangle.$$
- For pinned sites:
$$\langle x \mid x > 0 \rangle = \frac{\overline{\Delta^2}}{2\bar{\Delta}}.$$

Show that
$$v^2 + 2v(2F_c - F - 1) - 2\bar{\Delta}(F_c - F) = 0.$$

Exercise 12: Bienaymé–Galton–Watson branching process

At time $t = 0$ a single infected individual appears. Each infected individual:

- dies with rate a,
- branches with rate b.

On average,

$$R_0 = \frac{b}{a}.$$

Let $n(t)$ be the number of infected individuals at time t and

$$N(t) = \int_0^t n(t')dt'$$

the total number of infections.

We introduce

$$Q_s(t) = \int_0^\infty P(N)e^{-sN}dN = \left\langle e^{-s\int_0^t n(t')dt'} \right\rangle.$$

1. **Evolution equation**

   Show that

   $$Q_s(t+dt) = (1-(a+b)dt)e^{-sdt}Q_s(t) + a\,dt + b\,dt\,Q_s^2(t) + O(dt^2)$$

   and deduce

   $$\frac{dQ_s}{dt} = -(a+b+s)Q_s + a + bQ_s^2.$$

2. **Critical stationary solution**

   Set $a = b = 1$ and show that

   $$0 = -(2+s)Q_s^{\text{stat}} + 1 + (Q_s^{\text{stat}})^2$$

   with solution

   $$Q_s^{\text{stat}} = \frac{(2+s) - \sqrt{s^2+4s}}{2}.$$

   Show that

   $$Q_s^{\text{stat}} = 1 - \sqrt{s} + O(s).$$

3. **Critical asymptotics**

   Assume

   $$P(S) \sim AS^{-\tau}.$$

   Show that

   $$\tau = \frac{3}{2}, \qquad A = \frac{1}{2\sqrt{\pi}}.$$

Final result:

$$P(S) \sim \frac{1}{2\sqrt{\pi}}S^{-3/2}.$$