# PROBLEM 6

# Landscapes & Kac-Rice (2/2)

## Problem 6: The Hessian & Random Matrices

### $\boxed{1}$ GAUSSIAN RANDOM MATRICES

Consider GOE matrices with $P_N(M) = \frac{1}{Z_N} e^{-\frac{N}{4\sigma^2} \operatorname{tr} M^2}$.

Componentwise, this means:

$$P_N(\{M_{ij}\}_{i \leq j}) = \frac{1}{Z_N} e^{-\frac{N}{4\sigma^2} \sum_{ij} M_{ij}^2} = \frac{1}{Z_N} e^{-\frac{N}{4\sigma^2}\left[2 \sum_{i<j} M_{ij}^2 + \sum_i M_{ii}^2\right]}$$

Therefore, all the entries $M_{ij}$ with $i \geq j$ are independent and Gaussian, with zero mean and:

$$\overline{M_{ij}^2} = \frac{\sigma^2}{N} \quad \text{for } i \neq j$$

$$\overline{M_{ii}^2} = \frac{2\sigma^2}{N} \quad \text{for } i = j$$

This is exactly the same statistics as for the Hessian matrices of the p-spin landscape, with $\sigma^2 = p(p-1)$.

## [2] EIGENVALUE DENSITY & CONCENTRATION

- The determinant is the product of eigenvalues of a matrix. We denote with $\lambda_\alpha$, $\alpha = 1, \dots, N-1$ the eigenvalues of the matrix M. Notice: since the matrix has random entries, the eigenvalues are also random variables: they are a complicated, non-linear function of the entries of the matrix.

We can write:

$$|\det(M - p\varepsilon \mathbb{1})| = \prod_{\alpha=1}^{N-1} |\lambda_\alpha - p\varepsilon| = e^{\sum_{\alpha=1}^{N-1} \log|\lambda_\alpha - p\varepsilon|} =$$

$$= e^{\int d\lambda \sum_{\alpha=1}^{N-1} \delta(\lambda - \lambda_\alpha) \log|\lambda - p\varepsilon|}$$

$$= e^{(N-1)\int d\lambda\, \rho_{N-1}(\lambda) \log|\lambda - p\varepsilon|}$$

where we introduced the eigenvalue empirical distribution:

$$\rho_{N-1}(\lambda) = \frac{1}{N-1} \sum_{\alpha=1}^{N-1} \delta(\lambda - \lambda_\alpha)$$

- We now have to average this quantity on the distribution $P_N(M)$. However, we notice that this quantity depends on the matrix $M$ only through the eigenvalue density $\rho_{N-1}(\lambda)$. Therefore, we can make a change of variables and average over the distribution $P_N[\rho(\lambda)]$ of all possible eigenvalues densities:

$$\overline{|\det(M - p\varepsilon \mathbb{1})|} = \int dM \, P_N(M) \, |\det(M - p\varepsilon \mathbb{1})|$$

$$= \int \mathcal{D}\rho[\lambda] \, \underbrace{P_N[\rho(\lambda)]}_{\text{probability that } \rho_{N-1}(\lambda) = \rho(\lambda)} \, e^{N \int d\lambda \, \rho(\lambda) \log|\lambda - p\varepsilon| + o(N)}$$

functional integral

- We now use the fact that for $N$ large, $P_N[\rho(\lambda)]$ has a large-deviation form with speed $N^2$:

$$P_N[\rho] \sim e^{N^2 g[\rho]}$$

Therefore:

$$\overline{\det(M - p\varepsilon \mathbb{1})} = \int \mathcal{D}\rho(\lambda) \, e^{N^2 g[\rho(\lambda)] + N \int d\lambda \, \rho(\lambda) \log|\lambda - p\varepsilon| + o(N)}$$

This integral can be computed with the saddle-point approximation: the saddle-point value $\rho_\infty(\lambda)$ is determined by the leading-order term in the exponent, meaning:

$$\left.\frac{\delta g[\rho]}{\delta \rho}\right|_{\rho_\infty} = 0.$$

Moreover, one has that $g[\rho_\infty]=0$. Indeed, by normalization:

$$1 = \int \mathcal{D}\rho \, P_N[\rho] = \int \mathcal{D}\rho \, e^{N^2 g[\rho]} \underset{\substack{\uparrow \\ \text{saddle} \\ \text{point}}}{\sim} e^{N^2 g[\rho_\infty]} \Rightarrow g[\rho_\infty]=0$$

Therefore, $\rho_\infty(\lambda)$ is nothing but the typical value of the eigenvalue density, that maximizing the probability distribution:

$$\rho_\infty(\lambda) = \lim_{N\to\infty} \rho_N(\lambda)$$

If we know $\rho_\infty(\lambda)$, the expected value of the determinant is obtained as:

$$\overline{|\det(M - p\varepsilon \mathbb{1})|} = e^{N\int d\lambda \, \rho_\infty(\lambda) \log|\lambda - p\varepsilon| + o(N)}$$

# 3 THE SEMICIRCLE & THE COMPLEXITY

Combining everything from the previous problem 5, we obtain:

$$\overline{\mathcal{N}(\varepsilon)} = e^{\frac{N}{2}\log\left(\frac{e}{p}\right) - \frac{N}{2}\varepsilon^2 + N\int d\lambda\, \rho_\infty(\lambda)\log|\lambda - p\varepsilon| + o(N)}$$

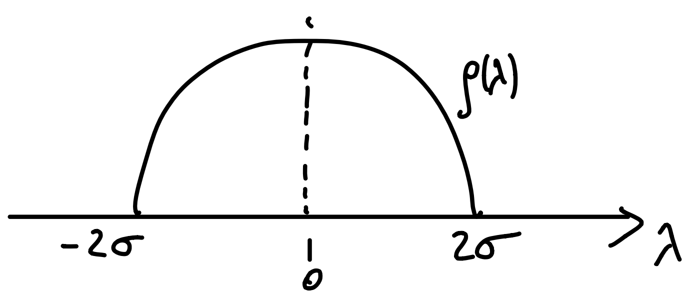
Therefore, the annealed complexity of the spherical p-spin is:

$$\Sigma_a(\varepsilon) = \lim_{N\to\infty} \frac{\log \overline{\mathcal{N}(\varepsilon)}}{N} = \frac{1}{2}\log\left(\frac{e}{p}\right) - \frac{\varepsilon^2}{2} + \int d\lambda\, \rho_\infty(\lambda)\log|\lambda - p\varepsilon| \quad (*)$$

To finish the calculation, one needs to know the expression of $\rho_\infty(\lambda)$.

■ Recall that $\rho_\infty(\lambda)$ is the eigenvalue density of the matrix $M$, that has GOE statistics.
It is a well known result of random matrix theory that the typical eigenvalue density of GOE matrices is the WIGNER SEMICIRCLE LAW:

$$\rho_\infty(\lambda) = \lim_{N\to\infty} \rho_N(\lambda) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - \lambda^2}$$



Plugging this expression into (*) and computing the integral, one obtains the final formulas given in the text of the problem. In particular:

$$\int d\lambda\, \rho_\infty(\lambda) \log|\lambda - p\varepsilon| = \int d\lambda \frac{1}{2\pi p(p-1)} \sqrt{4p(p-1) - (\lambda + p\varepsilon)^2}\, \log|\lambda|$$

$$= \int d\lambda \frac{\sqrt{4p(p-1)}}{2\pi p(p-1)} \sqrt{1 - \left(\frac{\lambda + p\varepsilon}{\sqrt{4p(p-1)}}\right)^2}\, \log|\lambda|$$

Define $x = \lambda / \sqrt{4p(p-1)}$

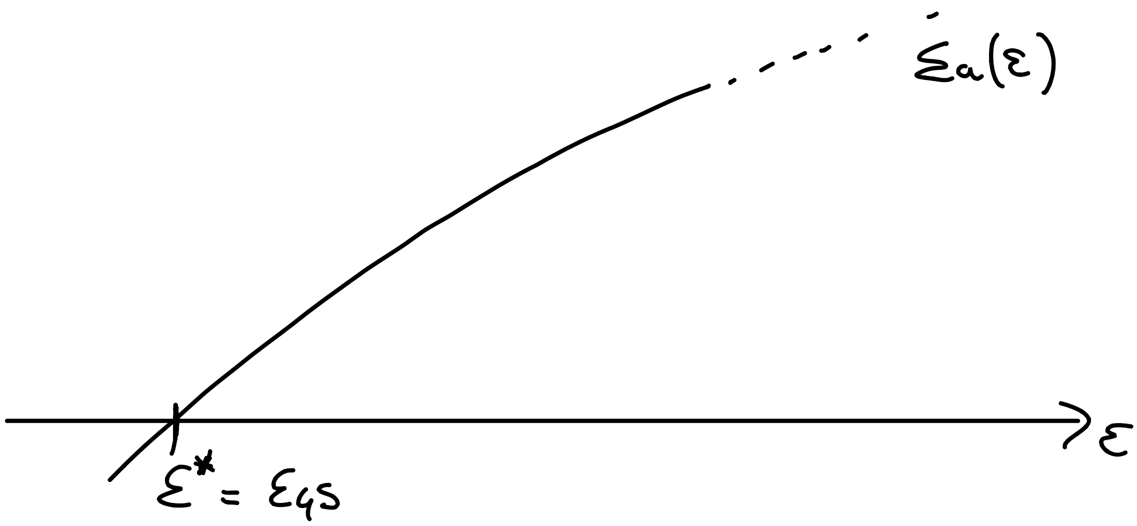
$$= \frac{2}{\pi} \int dx \sqrt{1 - \left(x - \frac{\varepsilon}{\varepsilon_m}\right)^2}\, \log\left|2\sqrt{p(p-1)}\, x\right|$$

$$= \frac{1}{2} \log\left[4p(p-1)\right] + \frac{2}{\pi} \int dx \sqrt{1 - \left(x - \frac{\varepsilon}{\varepsilon_m}\right)^2}\, \log|x|$$

where $\varepsilon_m = -2\sqrt{\frac{p-1}{p}}$.

The explicit result for the integral is given in the wiki.

III Plotting the function $\Sigma_a(\varepsilon)$ one gets:



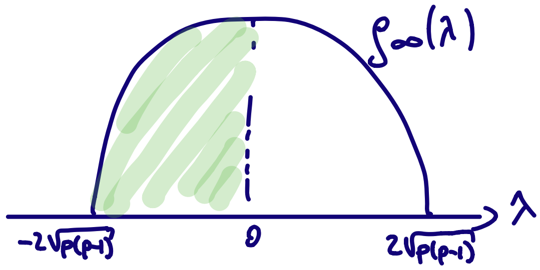

The region where the annealed complexity is negative is the region where it is exponentially unlikely to find local minima at that energy: the typical value of energy density of all local minima, including the deepest ones (the ground states) must be higher, i.e, $\varepsilon_{gs} \geq \varepsilon^*$.

Actually, for this model $\underset{\text{annealed}}{\Sigma_a(\varepsilon)} = \underset{\text{quenched}}{\Sigma(\varepsilon)}$, and $\varepsilon^* = \varepsilon_{gs}$.

# 4 THE THRESHOLD AND THE STABILITY

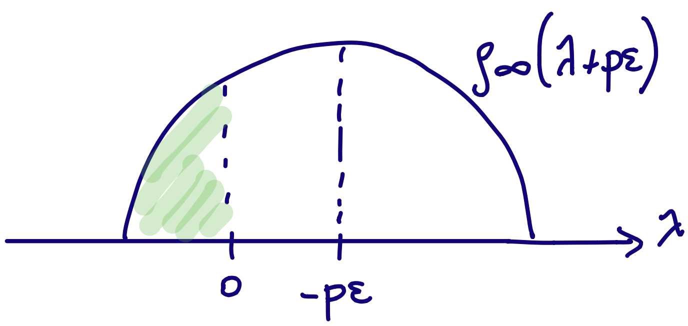
Recall that the Hessian matrix at a stationary point with energy density $\varepsilon$ has the statistics of $M - p\varepsilon \mathbb{1}$: if $\rho_\infty(\lambda)$ is the distribution of eigenvalues of $M$, the distribution of eigenvalues of $M - p\varepsilon\mathbb{1}$ is $\rho_\infty(\lambda + p\varepsilon)$. We discuss how this looks like changing $\varepsilon$.

$\boxed{\varepsilon = 0}$



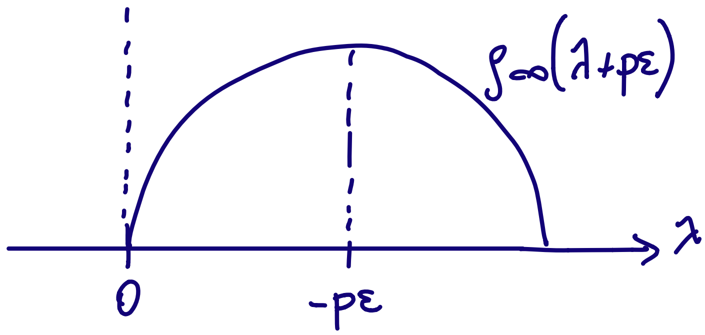

Half eigenvalues positive & half negative: SADDLE

$\boxed{\varepsilon < 0}$




Less negative eigenvalue, but still SADDLE

$\boxed{\varepsilon = \varepsilon_{th}}$ The boundary of the distribution touches zero:

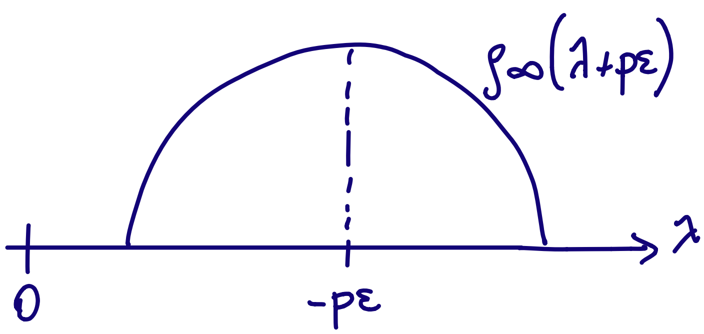



This happens when $-p\varepsilon = 2\sqrt{p(p-1)} \Rightarrow \varepsilon = \varepsilon_{th} = 2\sqrt{\frac{p-1}{p}}$

These type of minima are called

MARGINALLY STABLE

$\boxed{\varepsilon < \varepsilon_{th}}$

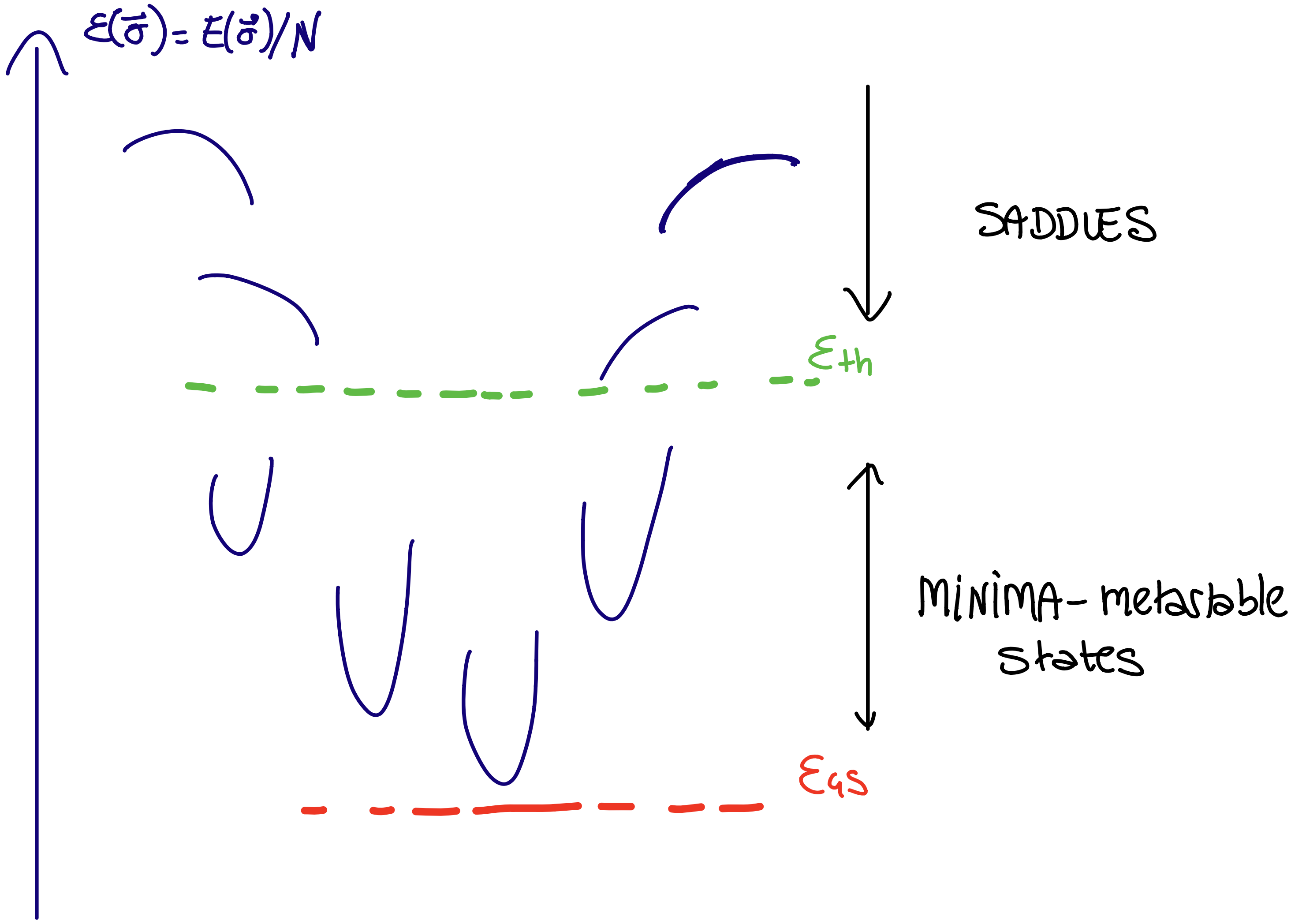



All eigenvalues are positive: STABLE MINIMA.

Minima for all energy densities in $[\varepsilon_{gs}, \varepsilon_{th}]$.

$\varepsilon > \varepsilon_{th}$ : Stationary points are SADDLES
$\varepsilon = \varepsilon_{th}$ : Stationary points are MARGINALLY STABLE MINIMA
$\varepsilon < \varepsilon_{th}$ : Stationary points are STABLE MINIMA

Stability transition in landscape at $\varepsilon = \varepsilon_{th}$:



# Comment: how to get $P_N[\rho]$?

- The functional $g[\rho]$ in $P_N[\rho]$ is the large-deviation functional for the eigenvalue density of GOE matrices.

  Its expression when $\sigma^2 = 1/2$ is:

$$g[\rho] = \frac{1}{2}\int d\lambda\, \lambda^2 \rho(\lambda) - \frac{1}{2}\int d\lambda\, d\lambda'\, \rho(\lambda)\rho(\lambda') \log|\lambda-\lambda'| + c\left(\int d\lambda\, \rho(\lambda) - 1\right)$$

  where $c$ is a Lagrange multiplier that enforces that

$$\int d\lambda\, \rho(\lambda) = 1.$$

  Optimizing this functional, one finds $\rho^*(\lambda) = \frac{1}{\sqrt{\pi}}\sqrt{2-\lambda^2}$ and $c = -(1+\log 2)/2$.

- One way to obtain $g[\rho]$ is through the joint eigenvalue density, which for GOE matrices is known explicitely:

$$\boxed{P_N(\lambda_1, \dots, \lambda_N) = \frac{1}{\tilde{Z}_N} e^{-N\sum_{\alpha=1}^{N}\frac{\lambda_\alpha^2}{4\sigma^2}} \prod_{\alpha<\beta} |\lambda_\alpha - \lambda_\beta|} \quad (**)$$

  This distribution is obtained from $P(M) = \frac{1}{Z} e^{-\frac{N}{4\sigma^2}\mathrm{tr}[M^2]}$ performing a change of variable, from the variables $\{M_{ij}\}_{i\le j}$ to the variables $\{\lambda_\alpha\}_{\alpha=1}^{N}$.

The term $\prod_{\alpha<\beta} |\lambda_\alpha - \lambda_\beta|$ is the Jacobian of the change of variables. It is called VANDERMONDE determinant.

It is the term that encodes the interactions between the eigenvalues of the random matix, in particular LEVEL REPULSION: the joint probability becomes small when two eigenvalues get close to each others.

- The distribution (**) can be interpreted as the partition function (with $\beta = 1$) of a gas of interacting particles:

$$\mathcal{P}_N(\lambda_1, .., \lambda_N) = \frac{1}{Z_N} e^{\frac{N}{4\sigma^2}\sum_{\alpha=1}^{N} \lambda_\alpha^2 + \sum_{\alpha\neq\beta} \log|\lambda_\alpha - \lambda_\beta|}$$

From here one can see that $P_N[\rho]$ can be obtained as:

$$P_N[\rho] = \frac{1}{Z_N} \int \prod_{\alpha=1}^{N} d\lambda_\alpha \, \mathcal{P}_N(\lambda_1, .., \lambda_\alpha) \, \delta\left[\rho - \frac{1}{N}\sum_{\alpha=1}^{N} \delta(\lambda - \lambda_\alpha)\right]$$

this is the starting point to get the expression for $g[\rho]$.

This is called COULOMB GAS FORMALISM.