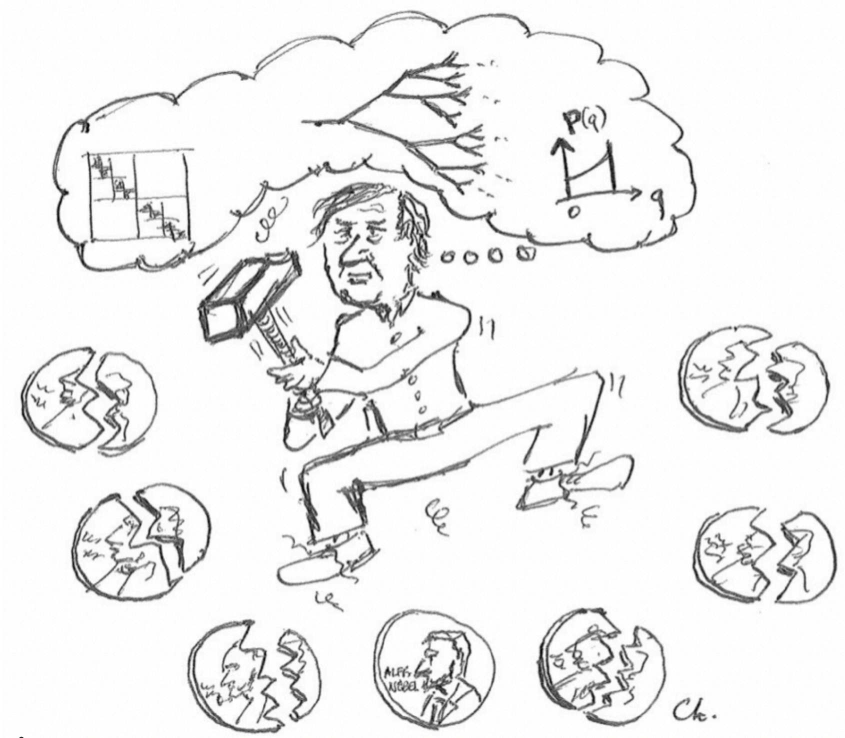


by Christophe Texier, LPTMS

prix Nobel de physique 2021 à Giorgio Parisi

SUMMARY LAST LECTURE

The saddle-point solutions (p-spin model)

$$\overline{Z_N^n} = \int \mathcal{D}Q \, e^{N A_n[Q] + o(N)} \qquad Q = \underbrace{\begin{pmatrix} 1 & q_{12} & \dots & q_{1n} \\ q_{12} & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}}_{\text{overlap matrix}}$$

Space of $n \times n$ matrices of overlaps $q_{ab} = \frac{\underline{\sigma}_a \cdot \underline{\sigma}_b}{N}$

Action: $$A_n[Q] = \frac{n}{2} \log(2\pi e) + \frac{1}{2} \log \det Q + \frac{\beta^2}{2} \sum_{a,b=1}^{n} q_{ab}^p$$

Saddle-point equations:

$$\frac{\delta A_n[Q]}{\delta q_{ab}} = \beta^2 p \, q_{ab}^{p-1} + (Q^{-1})_{ab} \Big|_{Q^*} = 0 \qquad a<b$$

- $\frac{n(n-1)}{2}$ coupled equations, to study when $n \to 0$
- infinite space where to search for solutions: need a good guess: "ansatz"
- if find more than one solution, choose the one with LOWER free-energy.

We mentioned two "ansätze":

(1) REPLICA SYMMETRIC

$$Q = \begin{pmatrix} 1 & q_0 & \cdots & q_0 \\ q_0 & 1 & q_0 & q_0 \\ & & \ddots & 1 \end{pmatrix} \xrightarrow[\text{when } n\to 0]{\text{saddle-point equation}} \beta^2 p\, q_0^{p-1} - \frac{q_0}{(1-q_0)^2}\Big|_{q_0^*} = 0$$

Solution: $q_0^* = 0 \Rightarrow Q = \mathbb{1}$

The free-energy coincides with annealed.

This is <u>always</u> a solution (for all T)

(2) 1-STEP REPLICA SYMMETRY BREAKING

$$Q = \begin{pmatrix} \boxed{\begin{matrix} 1 & q_1 \cdots & q_1 \\ & \ddots & 1 \end{matrix}} & q_0 & \\ & \boxed{\begin{matrix} 1 & & q_1 \\ & \ddots & \\ q_1 & & 1 \end{matrix}} & \\ & & \boxed{\begin{matrix} 1 & \ddots & q_1 \\ q_1 & & 1 \end{matrix}} \end{pmatrix}$$

$\longleftrightarrow$ size $\mu \leq n$

Three parameters: q_0, q_1, μ.

If you finish Problem 4.2 (solutions in Wiki!):

Solutions saddle-point equations ($n \to 0$):

- At high-T: only one solution, $\mu^* = 1$ and $q_1^* = 0 = q_0^*$
  This is the RS solution

- At $T = T_c$, another solution appears with
  $\mu^* = 1$, $q_0^* = 0$ and $q_1^* > 0$

  "RANDOM FIRST-ORDER TRANSITION"

- At $T < T_c$, good solution has
  $q_0^* = 0$, $\mu^*(T)$, $q_1^*(T)$ non-trivial T-dependence

  and $\lim_{T \to 0} q_1^*(T) = 1$, $\lim_{T \to 0} \mu^*(T) = 0$

Recall REM (Problem 2): $q_0^* = 0$, $q_1^* = 1$, $\mu^* = T/T_f$ for $T \leq T_f$

THE PHYSICS?

ORDERED PHASES: PURE STATES

- In "ordered phases", Boltzmann measure clusters into "PURE STATES" = sub-components $\alpha = 1, 2, \ldots$

  Each config. belongs only to one pure state.

- $A(\vec{\sigma})$ observable, $w_\alpha = \frac{1}{Z_N} \sum_{\vec{\sigma} \in \alpha} e^{-\beta E(\vec{\sigma})}$

  Then:

  $$\langle A \rangle = \sum_\alpha w_\alpha \langle A \rangle_\alpha$$

  average over Boltzmann

  average over pure state α

- Two configurations, e.g. $A(\vec{\sigma}, \vec{\sigma}') = q(\vec{\sigma}, \vec{\sigma}')$

  Then $\langle A(\vec{\sigma}, \vec{\sigma}') \rangle = \frac{1}{N} \sum_{i=1}^{N} \sum_{\gamma,\delta} w_\gamma w_\delta \langle \sigma_i \rangle_\gamma \langle \sigma_i' \rangle_\delta = \sum_{\gamma,\delta} w_\gamma w_\delta \cdot q_{\gamma\delta}$

$$P_\beta(q) = \langle \delta(q - q(\vec{\sigma}, \vec{\sigma}')) \rangle = \sum_{\gamma,\delta} w_\gamma w_\delta \, \delta(q - q_{\gamma\delta})$$

$$q_{EA} = q_{\alpha\alpha} = \lim_{N \to \infty} \frac{1}{N} \sum_i \langle \sigma_i \rangle_\alpha \langle \sigma_i \rangle_\alpha$$

Replicas probe the correlations between states!
The saddle-point captures them:

$$\overline{P_\beta(q)} = \lim_{n\to 0} \frac{2}{n(n-1)} \sum_{a<b} \delta\left(q - Q^*_{ab}\right)$$

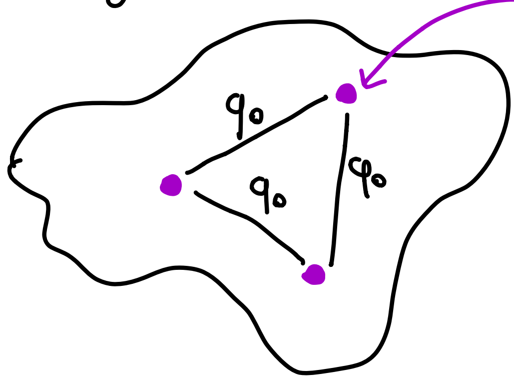
$$q_{EA} = \max_{a<b} \{ q^*_{ab} \}$$ – overlap between replicas <u>in same</u> state

HOW ENCODED IN REPLICA FORMALISM:

RS ANSATZ

$$Q = \begin{pmatrix} 1 & q_0 & q_0 & \\ q_0 & 1 & q_0 \cdots & q_0 \\ \vdots & & \ddots & \\ q_0 & \cdots & q_0 & 1 \end{pmatrix}$$

Only one state:



equilibrium configurations selected by Boltzmann measure ($N\to\infty$)

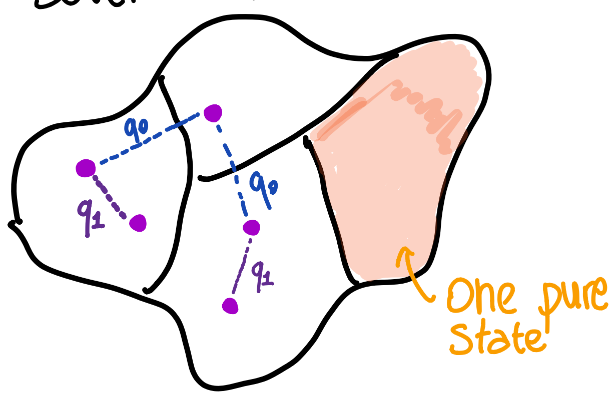
Spherical p-spin: solution $q_0^* = 0$ is good one at T high.

$$\begin{cases} \overline{P_\beta(q)} = \delta(q) \\ q_{EA} = 0 \end{cases} \Longrightarrow \text{PARAMAGNET}$$

1-RSB ANSATZ

$$Q = \begin{array}{c} \longleftarrow \mu \longrightarrow \qquad\qquad\qquad\qquad \\ \begin{pmatrix} \begin{matrix} 1 & & q_1 \\ & \ddots & \\ q_1 & & 1 \end{matrix} & q_0 & q_0 \\ & \begin{matrix} 1 & & q_1 \\ & \ddots & \\ q_1 & & 1 \end{matrix} & q_0 \\ & & \begin{matrix} 1 & & q_1 \\ & \ddots & \\ q_1 & & 1 \end{matrix} \end{pmatrix} \end{array}$$

Configurations selected by Boltzmann organize into several PURE STATES:



Spherical p-spin: at $T = T_c$, besides RS solution a 1RSB solution appears, with $\begin{cases} q_0^* = 0 \\ \mu^* = 1 \\ q_1^* > 0 \end{cases}$

This 1RSB solution is the <u>correct one</u> for $T \leq T_c$.

$$\begin{cases} \overline{P_\beta(q)} = \mu^*(T)\,\delta(q) + \left(1-\mu^*(T)\right)\delta\left(q - q_1^*(T)\right) & T \leq T_c \\ q_{EA} = q_1^*(T) \qquad \Longrightarrow \text{(SPIN) GLASS} & \end{cases}$$

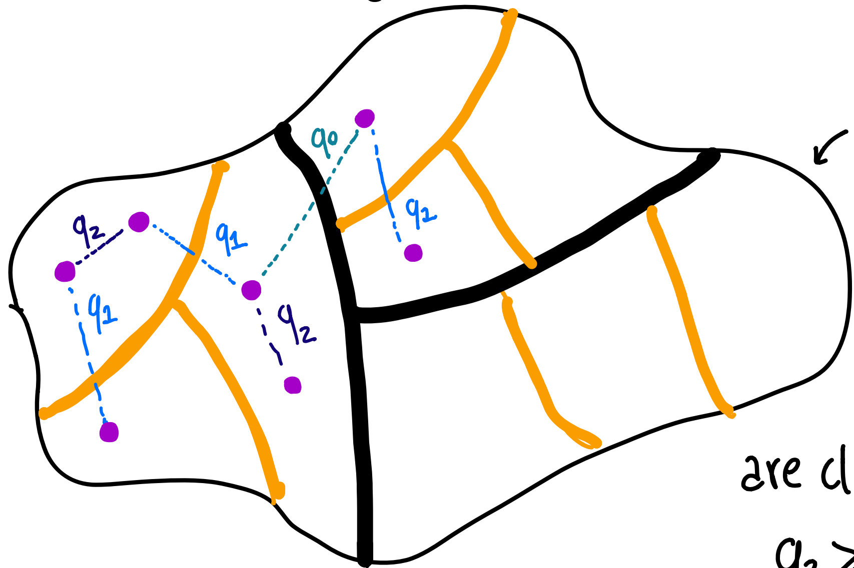
The 1RSB structure is exact for several models: spherical p-spin, constraint satisfaction problems... also structural glasses (mean-field)

K-RSB ANSATZ

[K=2]

$$Q = \begin{pmatrix} \begin{array}{c|c|c} \begin{matrix} 1 & q_2 \\ & \ddots \\ q_2 & 1 \end{matrix} & q_1 & q_1 \\ \hline q_1 & \begin{matrix} 1 & q_2 \\ & \ddots \\ q_2 & 1 \end{matrix} & q_1 \\ \hline q_1 & q_1 & \begin{matrix} 1 & q_2 \\ & \ddots \\ q_2 & 1 \end{matrix} \end{array} & & q_0 \\ & \boxed{\ddots} & \\ q_0 & & \begin{array}{c|c|c} \begin{matrix} 1 & q_2 \\ & \ddots \\ q_2 & 1 \end{matrix} & q_1 & q_1 \\ \hline q_1 & \begin{matrix} 1 & q_2 \\ & \ddots \\ q_2 & 1 \end{matrix} & q_1 \\ \hline q_1 & q_1 & \begin{matrix} 1 & q_2 \\ & \ddots \\ q_2 & 1 \end{matrix} \end{array} \end{pmatrix}$$

Clusters of states, $\boxed{\text{hierarchical organization}}$ of pure states: states inside of states...



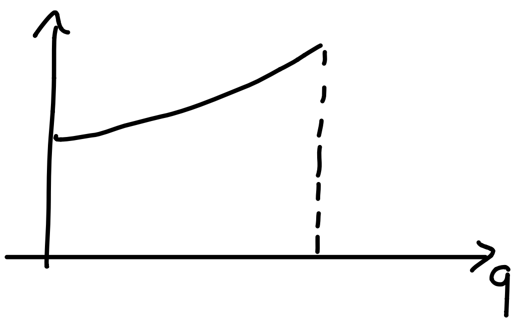
The $k=2$-step RSB ansatz describes situation in which pure states are clustered in groups: $q_2 \geq q_1 \geq q_0$

FULL-RSB ANSATZ

Iterate scheme an infinite number of times.

The overlap distribution becomes a continuous function.

$\overline{P_\beta(q)} =$

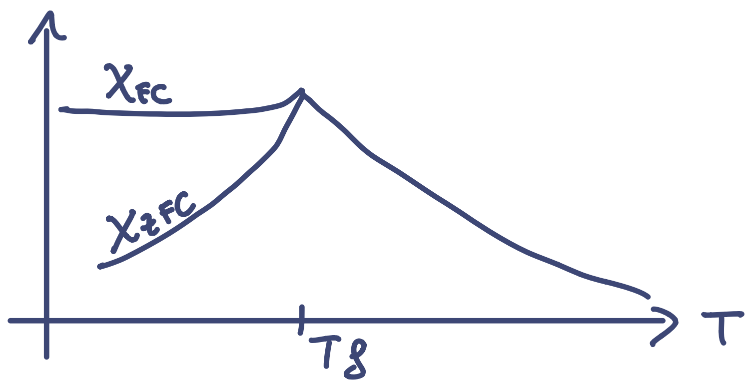


PROBING THESE SCENARIOS?

Recall two protocols for susceptibility:

(1) FIELD COOLED: add magnetic field, cool slowly the system with magnetic field, switch field to zero & measure response : χ_{FC}

(2) ZERO-FIELD COOLED: cool system, then add field, then switch to zero & measure response : χ_{ZFC}



In replica language

$$\chi_{FC} = \beta\left(1 - \int_0^1 dq\, \overline{P_\beta(q)}\, q\right)$$

measure response averaged over states

$$\chi_{ZFC} = \beta\left(1 - q_{EA}\right)$$

measure response within one state

FERROMAGNETS vs SPIN GLASSES

(Their equilibrium description)

FERROMAGNET	SPIN-GLASS

ORDER PARAMETER

Ferromagnet

$$m = \lim_{h\to 0} \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \langle \sigma_i \rangle_h$$

$$\langle \cdot \rangle_h = \frac{1}{Z_h} \sum_{\vec{\sigma}} e^{-\beta E(\vec{\sigma}) + \beta h \sum_{i=1}^{N} \sigma_i}$$

h - symmetry-breaking field

PARAMAGNET $(T \geq T_c)$: $m = 0$

FERROMAGNET $(T < T_c)$: $|m| > 0$

$(q_{EA} = m^2)$

Spin-glass

$$q_{EA} = \lim_{t\to\infty} \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \sigma_i(0)\sigma_i(t)$$

Thermodynamically?

$$q_{EA} = \lim_{\varepsilon\to 0} \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \langle \sigma_i^{(1)} \sigma_i^{(2)} \rangle_\varepsilon$$

$$\langle \ \rangle_\varepsilon = \frac{1}{Z_\varepsilon} \sum_{\vec{\sigma}^{(1)}, \vec{\sigma}^{(2)}} e^{-\beta E(\vec{\sigma}^{(a)}) + \varepsilon \beta \vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}}$$

small interaction

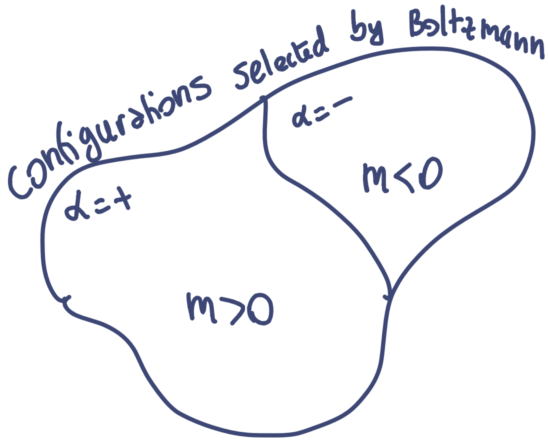
($\vec{\sigma}^{(1)}$ acts as a field for $\vec{\sigma}^{(2)}$)

PARAMAGNET $(T \geq T_c)$: $m = 0$, $q_{EA} = 0$

SPIN-GLASS $(T \leq T_c)$: $m = 0$, $|q_{EA}| > 0$

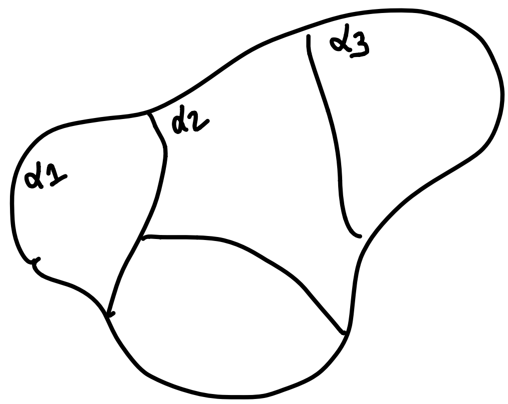
PURE STATES

Ferromagnet



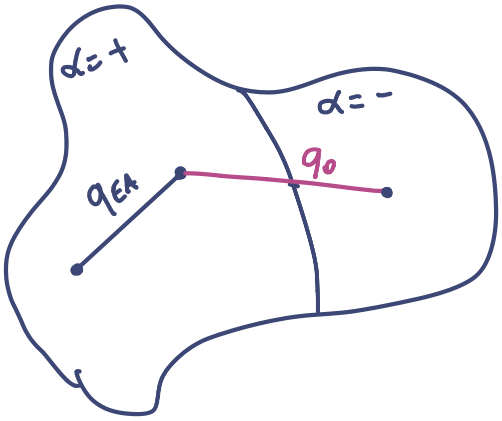
- Two pure states, $\alpha = \pm$
- Related by symmetry $(\mathbb{Z}_2)$
- Selected by field h

Spin-glass



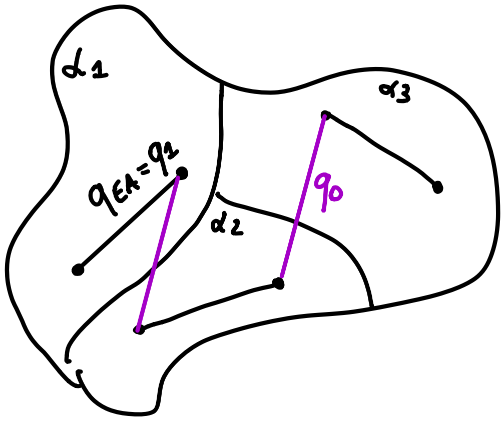
- More pure states
- Not related by symmetry
- To select one: couple to other equilibrium configuration

OVERLAPS



$$q_{EA} = m^2, \quad q_0 = -m^2$$

$$P_\beta(q) = \frac{1}{2}\left[\delta(q-m^2) + \delta(q+m^2)\right]$$



$q_{EA} = q_{\alpha\alpha}$ for all α

$$P_\beta(q) = \begin{cases} \text{sums of } \delta(\cdot) \text{ for K-RSB} \\ \text{continuous for full-RSB} \end{cases}$$

Always includes $q=0$.

BROKEN SYMMETRY

$\mathbb{Z}_2$

Replica symmetry (permutation symmetry when $n \to 0$)

COMMENTS

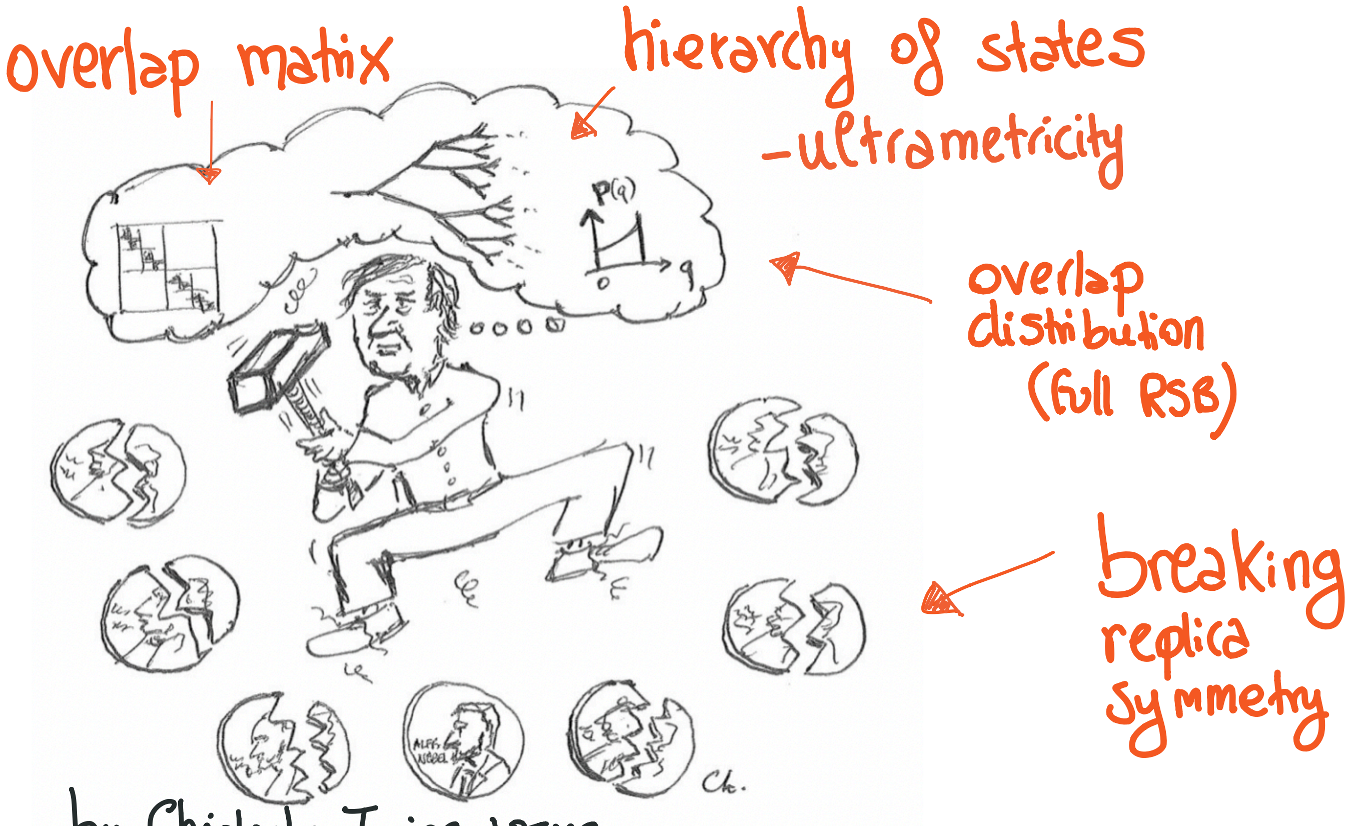
meaning, with hierarchical structure of states into states...

- How to break replica symmetry understood by G. PARISI in the 70s.
- Different RSB correspond to different physics (response to perturbations,...)
- The low-T solution of the SK model: $\sigma_i = \pm 1$

  $$E[\vec{\sigma}] = -\sum_{i<j} J_{ij}\, \sigma_i \sigma_j$$ is full-RSB.

  PARISI '79. Mathematically proven. GUERRA TALAGRAND PANCHENKO
- Mean-field models of structural glasses have a RS-1RSB transition ("random 1st order") and a 1RSB-full RSB transition ("Gardner transition").

  Solved $\simeq$ 2014: PARISI, URBANI, ZAMPONI
- In these days: replica theory used in several other disciplines, e.g., Machine Learning ...



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