SOLUTIONS TD8

Anderson on Bethe Lattice (1/2)

Problem 8 : Bethe Lattice, Recursions, Cavity

1 Green functions identities

By definition,

$$G = (z-H)^{-1} = \left[(z-H_0)\left(\mathbb{1} - (z-H_0)^{-1} V\right)\right]^{-1} = \left(\mathbb{1} - (z-H_0)^{-1} V\right)^{-1} G^0$$

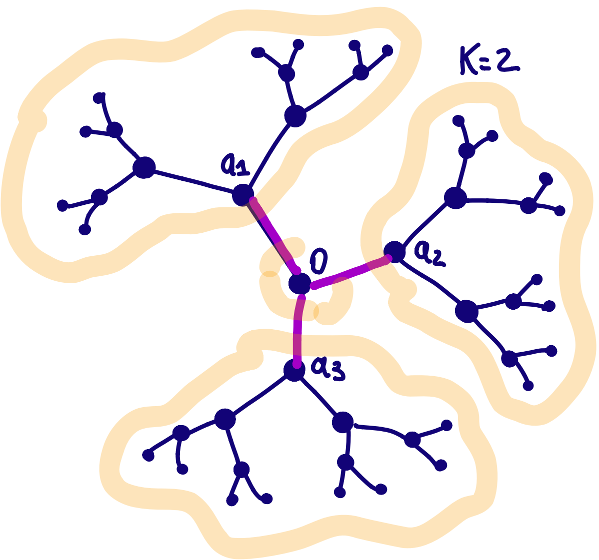
multiplying to the left by $\mathbb{1} - (z-H_0)^{-1} V$ we get:

$$(\mathbb{1} - G^0 V) G = G^0 \Rightarrow G - G^0 V G = G^0 \Rightarrow G = G^0 V G + G^0$$

When iterated, this relation gives rise to the perturbative series for G:

$$G = G_0 + G_0 V G_0 + G_0 V G_0 V G_0 + \cdots$$

[2] CAVITY EQUATIONS



In this case, the term V corresponds to the three links in pink in the figure.

Removing those links, one is decoupling the root from the $(K+1)$ subtrees with vertex $a_1, .., a_K$.

In particular, $H_0 = W\, \varepsilon_0 |0\rangle\langle 0| + \sum_{i=1}^{K} H_i^{(0)}$ where $H_i^{(0)}$ is the Hamiltonian restricted to the subtree with vertex a_i, $i=1,\dots,K+1$.
Since each subtree is completely disconnected with the root, the Green function $G_{a_i}^{cav}$ depends only on the Hamiltonian resticted to the subtree: it is the same that one would get removing the site 0.

The Green function relation can be iterated at any order.
Let us go to 2nd order in V:

$$G = G_0 + G_0 V G_0 + G_0 V G_0 V G$$

Let us take matrix elements:

$$G_{00} = G^0_{00} + \sum_{a,b} G^0_{0a} V_{ab} G^0_{b0} + \sum_{a,b,c,d} G^0_{0a} V_{ab} G^0_{bc} V_{cd} G_{d0}$$

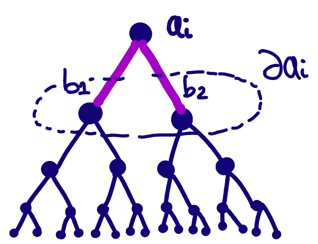
Now, G^0_{0a} is non zero only for $a=0$, and $V_{00}=0$: the first order vanishes. Also, V_{0a} is non-zero only if $a \in \{a_1, \dots, a_{K+1}\}$, the neighbors of 0, and it equals to $-V_{0a_i}$.

Thus:

$$G_{00} = G^0_{00} + \sum_{i=1}^{K+1} G^0_{00} V_{0a_i} G^0_{a_ia_i} V_{a_i0} G_{00} \Rightarrow G_{00} = \left(1 - G^0_{00} \sum_{i=1}^{K+1} V^2_{0a_i} G^{cav}_{a_i}\right)^{-1} G^0_{00}$$

Using that $G^0_{00} = (z-\varepsilon_0)^{-1}$, we get the first equation.

Let us iterate this procedure: we consider a subtree with origin in a_i, and define V the links connecting the origin to the "descendents":



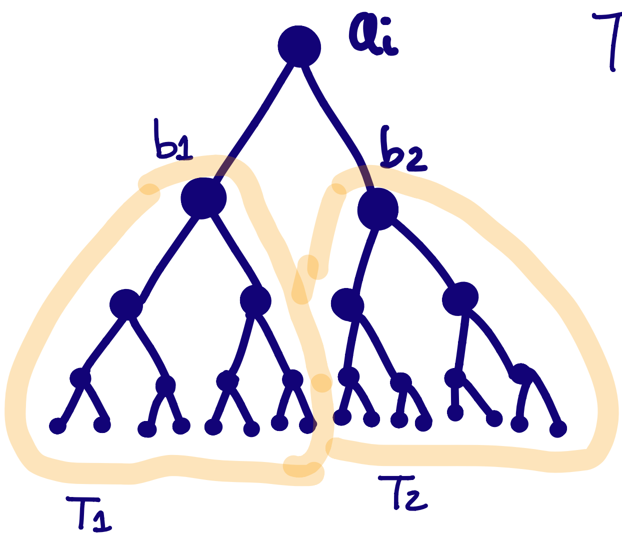
Repeating the steps above, we find:

$$G^{cav}_{a_i} = \frac{1}{z - \varepsilon_{a_i} - \sum_{b \in \partial a_i} V^2_{a_ib} G^{cav}_{b}} \overset{\text{by definition of } \sigma^{cav}}{=} \frac{1}{z - \varepsilon_{a_i} - \sigma^{cav}_{a_i}}$$

this sum is now over K sites, not K+1

$$\text{Then } \sigma^{cav}_{a_i} = \sum_{b \in \partial a_i} V^2_{a_ib} G^{cav}_b \overset{\text{use def of } \sigma^{cav}}{=} \sum_{b \in \partial a_i} V^2_{a_ib} \frac{1}{z - \varepsilon_b - \sigma^{cav}_b}$$

3 EQUATIONS FOR THE DISTRIBUTION



The functions $G^{cav}_{b_1}$ and $\sigma^{cav}_{b_1}$ depend only on the sites (and on the randomness ε_i) in the subtree T_1, which is not overlapping with the subtree T_2.

Thefore, the two random function are independent. Moreover, they are statistically equivalent (the sub-trees are statistically identical), and so they can be considered as identically distributed variables.

4 THE "LOCALIZED" SOLUTION

- We have $\sigma^{cav}_a = R_a - i\Gamma_a$

The cavity equation becomes:

$$R_{a_i}(E+i\eta) - i\Gamma_{a_i}(E+i\eta) = \sum_{b\in\partial a_i} V^2_{a_i b} \frac{1}{[E-\varepsilon_i - R_b(E+i\eta)] + i[\eta + \Gamma_b(E+i\eta)]}$$

$$= \sum_{b\in\partial a_i} V^2_{a_i b} \frac{(E-\varepsilon_i - R_b) - i(\eta + \Gamma_b)}{(E-\varepsilon_i - R_b)^2 - (\eta + \Gamma_b)^2}$$

Equating real and imaginary part, we get the two equations.

- The equation for Γ_a^2 is satisfied, for $\eta=0$, setting $\Gamma_a^2=\Gamma_b^2=0$.
  The solution $\Gamma^2=0$ corresponds to localization.
  It is always a solution when $\eta=0$.

- The Anderson criterion states that in the localized phase, when $\eta \downarrow 0$ the distibution of Γ^2 tends to $\delta(\Gamma^2)$.
  This means that the solution $\Gamma^2=0$, which holds for $\eta=0$, remains a stable solution also when adding $\eta>0$, taking $N\to\infty$ and then switching off η.

TO ESTABLISH LOCALIZATION, WE HAVE TO STUDY THE STABILITY OF THE SOLUTION $\mathbb{P}(\Gamma^2)=\delta(\Gamma^2)$.