Exercises 16-17: Trap model, and Localization

Valentina Ros, Alberto Rosso

Exercise 16. Connecting REM, localization, and trap model.

The goal of this exercise is to show that the trap model in Problem 7 is a good toy model to describe the dynamics of the REM, and also to draw some connections with the notion of localization. Consider again the REM model discussed in class: the energies E_α of the configurations $\alpha = 1, \cdots, M = 2^N$ are independent random variables with distribution $P(E)dE = (2\pi N)^{-1/2} e^{-\frac{E^2}{2N}} dE$. We recall that T_f is the REM freezing temperature.

Extreme values, again.

1. In the lectures, we have shown that the Ground State E_{min} has the statistics $E_{\text{min}} = E_{\text{min}}^{\text{typ}} + \frac{1}{\sqrt{2 \log 2}} z$, with z a Gumbel variable. Consider the extreme levels $E_\alpha = E_{\text{min}}^{\text{typ}} + \delta E_\alpha$, with $\delta E_\alpha \ll \sqrt{N}$. Show that the δE_α are exponentially distributed.
   Hint: approximate the Gumbel distribution for small argument, or use $(\delta E)^2/N \ll 1$.

2. Using point 1, compute the distribution of the variables

$$x_\alpha(\beta) = e^{-\beta\, \delta E_\alpha} \tag{1}$$

   and show that it is a power law

$$P_\mu(x)dx = \frac{c}{x^{1+\mu}} dx, \qquad c > 0. \tag{2}$$

   Determine the exponent μ, how it depends on T. Assume now $x > 1$: for which values of temperatures the first moment $\overline{x}$ and the second moment $\overline{x^2}$ exist?

Condensation and localization.

3. Consider now the REM partition function $Z = e^{-\beta E_{\text{min}}^{\text{typ}}} \sum_{\alpha=1}^{2^N} x_\alpha$. Using (2) and the dependence of μ on T, justify why for $T < T_f$ the sum Z is dominated by the extreme (maximal) x_α. Discuss in which sense this is consistent with the behavior of the REM free energy and entropy. In particular, intuitively, why can one talk about a condensation transition?

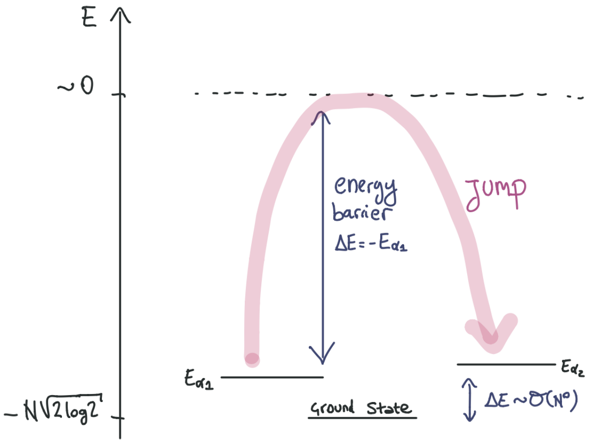
4. When computing the overlap distribution of the REM we have introduced the quantity

$$I_2^{(N)}(\beta) = \frac{\sum_{\alpha=1}^{2^N} z_\alpha^2}{[\sum_{\alpha=1}^{2^N} z_\alpha]^2} = \frac{\sum_{\alpha=1}^{2^N} x_\alpha^2}{[\sum_{\alpha=1}^{2^N} x_\alpha]^2} = \sum_{\alpha=1}^{2^N} \omega_\alpha^2, \qquad z_\alpha = e^{-\beta E_\alpha}, \qquad \omega_\alpha = \frac{x_\alpha}{\sum_{\beta=1}^{2^N} x_\beta}. \tag{3}$$

   One can show that that if the x_α have the distribution (2), the normalized weights ω_α have the distribution:

$$\tilde{P}_\mu(\omega) = \frac{\omega^{-(1+\mu)}(1-\omega)^{\mu-1}}{\Gamma(\mu)\Gamma(1-\mu)}, \tag{4}$$

   where $\Gamma(z)$ is the Gamma function. Compute $\lim_{N\to\infty} \mathbb{E}[I_2^{(N)}(\beta)]$ and recover the result for the REM discussed in class.



5. Using what we have discussed in the lectures about localization, justify why this quantity plays the same role as the Inverse Participation Ratio (IPR), and explain in which sense the low-T phase of the REM is analogous to a localized phase.

From the REM to the Trap model.

5. Consider now a Metropolis dynamics at small temperature T, and consider processes in which the system transitions between two of these configurations with extreme energy, passing through intermediate configuration with energy density $\epsilon = 0$: $E_{\alpha_1} \to 0 \to E_{\alpha_2}$, see Fig. . Using the Arrhenius law and the results above, compute the form of the probability distribution of these transition times (or trapping times) and recover the trap model discussed in class.
   Hint. Use the Arrhenius law, that states that the typical time needed for a transition to occur scales as $\tau \sim e^{\beta \Delta E}$, where ΔE is the energy barrier between the initial and final the configurations.

6. Explain why the non-ergodic phase of the dynamics of the trap model (occurring at $\mu < 1$) can be interpreted as a description of a REM that tries to equilibrate at $T < T_f$.

Exercise 17. Weak disorder expansion in the 1D Anderson model

Consider the one-dimensional Anderson model

$$-\psi_{n+1} - \psi_{n-1} + V_n \psi_n = \epsilon \psi_n, \tag{5}$$

where the random potentials V_n are independent random variables with

$$\overline{V_n} = 0, \qquad \overline{V_n^2} = W^2. \tag{6}$$

Assume weak disorder $W \ll 1$.

1. In the absence of disorder, show that the eigenstates are plane waves

   $$\psi_n = e^{ikn} \tag{7}$$

   with dispersion relation

   $$\epsilon = -2\cos k. \tag{8}$$

2. Introduce the Riccati variable

$$z_n = \frac{\psi_n}{\psi_{n-1}}. \tag{9}$$

Show that it satisfies the recursion relation

$$z_{n+1} = V_n - \epsilon - \frac{1}{z_n}. \tag{10}$$

What is the value of z_n in the clean system?

3. Show that

$$\psi_N = \prod_{n=1}^{N} z_n, \tag{11}$$

and deduce that the Lyapunov exponent can be written as

$$\gamma = \lim_{N\to\infty} \frac{1}{N} \overline{\ln|\psi_N|} = \overline{\ln|z_n|}, \tag{12}$$

assuming stationarity of $\overline{\ln|z_n|}$. Note that once more we need the average of $\ln|z_n|$ rather than the average of $|z_n|$.

4. For weak disorder we write

$$z_n = e^{ik}(1+\eta_n), \qquad |\eta_n| \ll 1. \tag{13}$$

Write

$$\eta_n = x_n + iy_n, \tag{14}$$

with $x_n, y_n \in \mathbb{R}$. Show that, to second order in η_n,

$$\ln|z_n| = x_n - \frac{x_n^2}{2} + \frac{y_n^2}{2} + O(\eta_n^3). \tag{15}$$

Deduce that

$$\overline{\ln|z_n|} = \mathrm{Re}\,\overline{\eta_n} - \frac{1}{2}\mathrm{Re}\,\overline{\eta_n^2} + O(W^3). \tag{16}$$

5. We now compute the moments of η_n perturbatively.

Substitute

$$z_n = e^{ik}(1+\eta_n) \tag{17}$$

into the Riccati recursion and keep the parametrization of the energy used in the clean system,

$$\epsilon = -2\cos k, \tag{18}$$

even though in the presence of disorder k is no longer a good quantum number but only a convenient parametrization of the energy. Expand to second order in η_n. Show that

$$\eta_{n+1} = e^{-ik}V_n + e^{-2ik}\eta_n - e^{-2ik}\eta_n^2 + O(W^3). \tag{19}$$

Hint: use

$$\frac{1}{1+\eta_n} \simeq 1 - \eta_n + \eta_n^2. \tag{20}$$

6. Keeping only the linearized recursion,

$$\eta_{n+1}^{(1)} = e^{-ik}V_n + e^{-2ik}\eta_n^{(1)}, \tag{21}$$

we first determine the stationary solution of this recursion.

(a) Show by iterating the recursion that

$$\eta_n^{(1)} = e^{-ik} \sum_{m=1}^{\infty} e^{-2ik(m-1)} V_{n-m}. \tag{22}$$

(b) Using the independence of the disorder variables,

$$\overline{V_n V_m} = W^2 \delta_{nm}, \tag{23}$$

compute $\overline{\left(\eta_n^{(1)}\right)^2}$.

(c) Show that

$$\overline{\left(\eta_n^{(1)}\right)^2} = W^2 \, e^{-2ik} \sum_{m=0}^{\infty} e^{-4ikm}. \tag{24}$$

(d) Evaluate the geometric series and show that

$$\overline{\left(\eta_n^{(1)}\right)^2} = -\frac{iW^2}{2\sin(2k)}. \tag{25}$$

Hint: you may use

$$1 - e^{-4ik} = 2i \, e^{-2ik} \sin(2k). \tag{26}$$

Deduce in particular that

$$\mathrm{Re}\, \overline{\eta_n^2} = 0. \tag{27}$$

7. Average the full recursion relation obtained in question 5:

$$\overline{\eta_{n+1}} = e^{-2ik} \, \overline{\eta_n} - e^{-2ik} \, \overline{\eta_n^2} + O(W^3). \tag{28}$$

Assume that the moments of η_n have reached a stationary limit, so that

$$\overline{\eta_{n+1}} = \overline{\eta_n} \equiv \mu. \tag{29}$$

Using the result of question 6, deduce that

$$\mathrm{Re}\, \mu = \frac{W^2}{8 \sin^2 k}. \tag{30}$$

Hint: use the identities

$$1 - e^{-2ik} = 2i \, e^{-ik} \sin k, \qquad \sin(2k) = 2 \sin k \cos k. \tag{31}$$

8. Using the previous questions, deduce that

$$\gamma(\epsilon) = \frac{W^2}{8 \sin^2 k}. \tag{32}$$

Hence obtain the localization length

$$\xi_{\mathrm{loc}} = \gamma^{-1} \tag{33}$$

and express it as a function of the energy ϵ.

9. Discuss briefly why this perturbative result cannot be trusted near the band edges $\epsilon \simeq \pm 2$, and why the band center $\epsilon = 0$ is also a special point.