Exercises 5-6: Random Energy Models

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Notation throughout the exercises. We denote with Z the partition function of the model, with $\beta = 1/T$ the inverse temperature, with $\langle \cdot \rangle$ the average over the Boltzmann measure and with $\bar{\cdot}$ the average over the randomness.

Exercise 5: A diluted Random Energy Model (REM)

Consider a model with $M = 3^N$ configurations $\vec{\sigma}^\alpha = (\sigma_1^\alpha, \cdots, \sigma_N^\alpha)$, where $\alpha = 1, \cdots, M$ and $\sigma_i^\alpha \in \{0, +1, -1\}$. The energy of each configuration is

$$E(\boldsymbol{\sigma}^\alpha) = \sqrt{Q(\vec{\sigma}^\alpha)}\, X_\alpha + \Delta\, Q(\vec{\sigma}^\alpha), \qquad Q(\vec{\sigma}^\alpha) = \sum_{i=1}^N [\sigma_i^\alpha]^2,$$

where X_α are independent Gaussian random variables with zero mean and unit variance, and Δ is a real parameter.

1. Explain why the distribution of the energy levels of this model is

$$P_\Delta(E) = \frac{1}{3^N} \sum_{Q=0}^N 2^Q \binom{N}{Q} c(Q)\, e^{-\frac{(E-\Delta\, Q)^2}{2Q}}, \qquad \binom{N}{Q} := \frac{N!}{Q!(N-Q)!} \tag{1}$$

   and compute $c(Q)$.

2. Let $\mathcal{N}(E)dE$ be the number of configurations in the interval $[E, E+dE]$. Explain how to get the annealed free energy from the average $\overline{\mathcal{N}(E)}dE$. Introduce the densities $\epsilon = E/N$ and $q = Q/N$ and show that for N large one can write

$$\overline{\mathcal{N}(E)} \approx \int_0^1 dq\, e^{N f(q,\epsilon,\Delta)}, \tag{2}$$

   and determine the function $f(q, \epsilon, \Delta)$.
   Hint. Use the asymptotic formula $\log x! \sim x \log x - x$.

3. Justify why the annealed entropy of the model equals to

$$S_a(\epsilon, \Delta) = -\frac{\epsilon^2}{q_*} + \epsilon\, \Delta - \log(1 - q_*), \tag{3}$$

   where $q_* = q_*(\epsilon, \Delta)$ solves the equation

$$\log\left(\frac{2(1-q)}{q}\right) + \frac{(\epsilon - \Delta\, q)^2}{2q^2} + \frac{\Delta}{q}(\epsilon - \Delta\, q) = 0.$$

   Let $[\epsilon_-(\Delta), \epsilon_+(\Delta)]$ be the interval where $S_a(\epsilon, \Delta) \geq 0$: in analogy with the standard REM, what is the quenched entropy? Where and why quenched and annealed entropy differ?

4. Derive the equation for the equilibrium energy density ϵ_β at inverse temperature $\beta = 1/T$. Using such equation and the two relations

$$|\epsilon| = q_* \sqrt{\Delta^2 - 2\log\left(\frac{2(1-q_*)}{q_*}\right)}, \qquad \frac{dq_*}{d\epsilon} = \frac{\epsilon/q_*}{\frac{\epsilon^2}{q_*^2} + \frac{1}{1-q_*}}, \tag{4}$$

show that

$$\epsilon_\beta = q_* (\Delta - \beta), \qquad q_* = \frac{2}{2 + e^{\beta\Delta - \frac{\beta^2}{2}}} \tag{5}$$

in the high temperature phase. When does this solution cease to be the correct one? Write an implicit equation for the freezing inverse temperature $\beta_f(\Delta)$.

5. Tuning the value of Δ, one can favor energetically configurations with higher or smaller values of Q: for which limiting value of Δ you expect to recover the standard REM with $\sigma_i^\alpha = \pm 1$? Show that in this limit for Δ, the entropy $S_a(\epsilon_\beta, \Delta)$ of the equilibrium configurations converges to the one we have computed for the standard REM, and recover from there the REM freezing temperature. As a function of Δ, $\beta_f(\Delta)$ is monotonic: do you expect it to be decreasing, or increasing with Δ? Why?

Exercise 6. A Random Energy Model with generalized energy distribution

Consider a model with $M = 2^N$ configurations with random independent energies E_i distributed according to:

$$P_\delta(E) = C\, e^{-A(N)|E|^\delta}, \qquad C,\ A(N),\ \delta > 0. \tag{6}$$

1. Determine the normalization constant C when $\delta = 1$. For which values of A, δ does this model reduce to the one discussed in class?

2. Using extreme value statistics, determine the typical value of the Ground State energy $E_{GS}^{\rm typ}$. Set $A(N) = A \cdot N^\alpha$: which choice of α guarantees that the ground state is an extensive quantity? Recover the results obtained in the case of the REM discussed in class.

<u>*The case $\delta > 1$.*</u> Consider now $\delta > 1$, and $A(N) = A \cdot N^\alpha$ with the exponent α determined in point 2.

3. Compute the function $g_\delta(\lambda)$ defined as:

$$\int_{-\infty}^{\infty} dE\, P_\delta(E)\, e^{-\lambda E} \equiv e^{N g_\delta(\lambda) + o(N)} \qquad \lambda > 0. \tag{7}$$

Hint: use that $\frac{d}{dx}|x| = \text{sign}(x)$ where $\text{sign}(x) = 1$ if $x > 0$, and $\text{sign}(x) = -1$ otherwise.

4. Using point 3, compute the average partition function $\overline{Z}$ and the annealed free-energy density

$$f_a(T) = -T \lim_{N\to\infty} \frac{\log \overline{Z}}{N}. \tag{8}$$

Show that the annealed entropy density $s_a(T) \equiv -\frac{df_a(T)}{dT}$ equals to:

$$s_a(T) = \log 2 - A[\delta\, A\, T]^{-\frac{\delta}{\delta-1}}. \tag{9}$$

Determine the freezing temperature T_f of the model for general δ. What do you expect to be the behavior of the free energy density obtained within the quenched formalism for both $T > T_f$ and $T < T_f$?
Hint: to solve this point, you can use the notion of entropy crisis.

5. Within a replica calculation with a 1RSB ansatz, one can show that

$$\overline{Z^n} = \int_0^1 d\mu \, M^{\frac{n}{\mu}} e^{N \frac{n}{\mu} g_\delta\left(\frac{\mu}{T}\right) + o(Nn)} \tag{10}$$

where the order parameter μ has the same meaning as that discussed in the lectures. Let us denote with μ^* the saddle-point value of μ, which optimizes the function at the exponent in (10).

A. Using (10) and the replica trick, show that the quenched free energy is:

$$f_q(T) = -T \lim_{N\to\infty} \frac{1}{N} \overline{\log Z} = f_a\left(\frac{T}{\mu^*}\right). \tag{11}$$

B. Write the saddle point equation for μ. Using this equation and the fact that $s_a(T) \equiv -\frac{df_a(T)}{dT}$, show that μ^* satisfies

$$\frac{1}{[\mu^*]^2} \, s_a\left(\frac{T}{\mu^*}\right) = 0. \tag{12}$$

Determine μ^* (hint: use the definition of T_f also used in point 2). Is this consistent with what was discussed in the lecture for the standard REM?

<u>*The case $\delta \le 1$.*</u> Consider now $\delta \le 1$, and $A(N) = A \cdot N^\alpha$ with the exponent α determined in point 1 and $A = 1$. For $\delta \le 1$, we can not generalize the replica approach above.

6. One can show that the typical value of the partition function can be written as

$$Z^{\text{typ}} = \int_{\epsilon_{GS}^{\text{typ}}}^{0} d\epsilon \, e^{N[S(\epsilon) - \beta\epsilon] + o(N)} \qquad S(\epsilon) = \log 2 - |\epsilon|^\delta, \tag{13}$$

where $\epsilon_{GS}^{\text{typ}} < 0$ is the typical value of the ground state energy density determined in point 1. Compute the saddle point value ϵ^* of the energy density, and use it to recover the freezing temperature T_f obtained in point 4. Check that at $T = T_f$, $\epsilon^* = \epsilon_{GS}^{\text{typ}}$.

7. Compute Z^{typ} in the special case $\delta = 1$: show that the saddle-point value for ϵ jumps at $T_f = 1$. How does the entropy behave at the transition? Why can one say that the transition is in a different universality class with respect to the case $\delta > 1$?