PROBLEMS 4

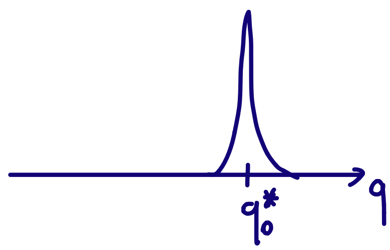
The replica method (2/2)

Problem 4.1: The RS Calculation

1 The RS overlap distribution

Under the RS assumption, the overlap distribution is simply:

$$\overline{P_\beta(q)} = \delta(q - q_0^*)$$



The overlap can take only one value, that must coincide with the overlap between configurations in the same pure state, which is therefore unique. Also, $q_{EA} = q_0^*$.

2 RS free-energy

$$Q^{-1} = \begin{pmatrix} a & b & b & b \\ b & a & & b \\ b & b & a \ddots & b \\ b & b & b \ddots & a \end{pmatrix}$$

Same RS structure as Q

To determine a and b, impose $Q \cdot Q^{-1} = \mathbb{1}$.

We have:

$$Q \cdot Q^{-1} = \begin{pmatrix} c_1 & c_2 & \cdots & c_2 \\ c_2 & c_1 & & \\ \vdots & & \ddots & \\ c_2 & & & c_1 \end{pmatrix}$$

where

$$c_1 = a + b q_0 (n-1) = \frac{1 + (n-2) q_0 - (n-1) q_0^2}{1 + (n-2) q_0 - (n-1) q_0^2} = 1$$

$$c_2 = a q_0 + b + (n-2) q_0 b = \frac{q_0 + (n-2) q_0^2 - q_0 - (n-2) q_0^2}{1 + (n-2) q_0 - (n-1) q_0^2} = 0$$

$$\Rightarrow \begin{cases} a = \dfrac{1 + (n-2) q_0}{[1 + (n-2) q_0 - (n-1) q_0^2]} \\ b = \dfrac{-q_0}{[1 + (n-2) q_0 - (n-1) q_0^2]} \end{cases}$$

The saddle-point equation reads:

$$\beta^2 p q_0^{p-1} - \frac{q_0}{1 + (n-2) q_0 - (n-1) q_0^2} = 0$$

$$(n \to 0) \Rightarrow \beta^2 p q_0^{p-1} - \frac{q_0}{(1-q_0)^2} \Bigg|_{q_0^*} = 0$$

Which is solved by $q_0^* = 0$. This is the paramagnetic solution: the typical overlap between two equilibrium configurations is zero, meaning that the magnetization patterns are uncorrelated.

In problems 3 we got:

$$\overline{Z^n} = \int \prod_{a<b} dq_{ab} \; e^{N A_n[Q] + o(N)}$$

with $\quad A_n[Q] = \frac{n}{2}\log(2\pi e) + \frac{1}{2}\log\det Q + \frac{\beta^2}{2}\sum_{a,b} q_{ab}^p$

If $q_0^* = 0$, then $Q^* = \mathbb{1}$ and

$$\overline{Z^n} = e^{\frac{Nn}{2}\log(2\pi e) + \frac{Nn}{2}\beta^2 + o(N)}$$

and using the replica trick:

$$f = \lim_{N\to\infty} \lim_{n\to 0} \frac{-1}{\beta}\left(\frac{\overline{Z^n} - 1}{Nn}\right) = -\frac{1}{\beta}\left[\frac{\log(2\pi e)}{2} + \beta^2/2\right] = f_a$$

The RS free-energy coincides with the annealed.

Problem 4.2: The RSB Calculation

[1] THE 1RSB OVERLAP DISTRIBUTION

We have $\overline{P_\beta(q)} = \lim_{n\to 0} \frac{2}{n(n-1)} \sum_{a>b} \delta(q - q^*_{ab})$

In the 1-RSB ansatz with parameters μ^*, q_0^*, q_1^*:

$$\sum_{a>b} \delta\left(q - q^*_{ab}\right) = \frac{n}{\mu^*} \, \frac{\mu^*(\mu^*-1)}{2} \, \delta\left(q - q_1^*\right) +$$

number of diagonal blocks of Q

number of off-diagonal elements in each block

$$+ \left[\frac{n(n-1)}{2} - \frac{n}{\mu^*} \frac{\mu^*(\mu^*-1)}{2}\right] \delta\left(q - q_0^*\right)$$

$$\overline{P_\beta(q)} = \lim_{n\to 0} \left[\frac{\mu^*-1}{n-1} \delta\left(q - q_1^*\right) + \left(\frac{n-\mu^*}{n-1}\right) \delta\left(q - q_0^*\right)\right]$$

$$= (1-\mu^*)\, \delta\left(q - q_1^*\right) + \mu^* \, \delta\left(q - q_0^*\right)$$

Therefore, the overlap distribution now has two peaks: one which corresponds to the overlap within one state, and one with the overlap between replicas falling in different states. Like in REM.

The quantity $(1-\mu^*)$ gives the probability that extracting two configurations at equilibrium, they are found in the same pure state.
In the REM, we got $q_1^* \to 1$, $q_0^* \to 0$ and $\mu^* = T/T_f$ for $T \leq T_f$. In the spherical p-spin, these parameters have to be fixed by the saddle point equations.

[2] 1RSB FREE-ENERGY & SADDLE POINT EQUATIONS

The expression of the 1RSB free-energy is derived below.
The RS limit is obtained when $\mu \to 1$.

Let us derive the saddle point equations.

- EQUATION FOR q_0

$$\frac{\partial f_{1RSB}}{\partial q_0} = \frac{-1}{2\beta}\left[-\mu p \beta^2 q_0^{p-1} - \frac{1}{\mu(q_1-q_0)+1-q_1} + \frac{[\mu(q_1-q_0)+1-q_1]+q_0\mu}{[\mu(q_1-q_0)+1-q_1]^2}\right]$$

$$= \frac{-1}{2\beta}\left[-\mu p \beta^2 q_0^{p-1} + \frac{q_0\mu}{[\mu(q_1-q_0)+1-q_1]^2}\right] = 0$$

This admits the solution $q_0^* = 0$.

- <u>EQUATION FOR q_1</u>

$$\frac{\partial f_{1RSB}}{\partial q_1} = -\frac{1}{2\beta}\left\{\beta^2 p(\mu-1)q_1^{p-1} - \frac{(\mu-1)}{\mu}\frac{1}{1-q_1} + \frac{(\mu-1)}{\mu[\mu(q_1-q_0)+1-q_1]} + \right.$$

$$\left. - \frac{q_0(\mu-1)}{[\mu(q_1-q_0)+1-q_1]^2}\right\} = 0$$

For $q_0 = 0$ this becomes:

$$\beta^2 p(\mu-1)q_1^{p-1} - \frac{\mu-1}{\mu}\frac{1}{1-q_1} + \frac{(\mu-1)}{\mu[1+(\mu-1)q_1]} = 0 \qquad (1)$$

- <u>EQUATION FOR μ</u>

$$\frac{\partial f_{1RSB}}{\partial \mu} = -\frac{1}{2\beta}\left\{\beta^2(q_1^p - q_0^p) + \frac{1}{\mu^2}\log\left(\frac{1-q_1}{1-q_1+\mu(q_1-q_0)}\right) + \right.$$

$$\left. + \frac{1}{\mu}\frac{(q_1-q_0)}{[1-q_1+\mu(q_1-q_0)]} - \frac{(q_1-q_0)q_0}{[1-q_1+\mu(q_1-q_0)]^2}\right\} = 0$$

For $q_0 = 0$:

$$\beta^2 q_1^p + \frac{1}{\mu^2}\log\left(\frac{1-q_1}{1+(\mu-1)q_1}\right) + \frac{q_1}{\mu[1+(\mu-1)q_1]} = 0 \qquad (2)$$

3 THE "RANDOM FIRST ORDER" TRANSITION

The 1RSB saddle point equations always admit the solution $\mu^* = 1$, $q_0^* = 0 = q_1^*$: the paramagnet.

However, when $T \leq T_c$ a second solution appears, which has a lower free-energy. We assume that μ^* is continuous at $T = T_c$, meaning that $\mu^* = 1$ also at T_c. Then equation (1) is satisfied, and equation (2) becomes:

$$\beta^2 q_1^p + \log(1 - q_1) + q_1 = 0$$

One can study this equation graphically for various β and p.

Let $F(q_1) = \beta^2 q_1^p + \log(1-q_1) + q_1$

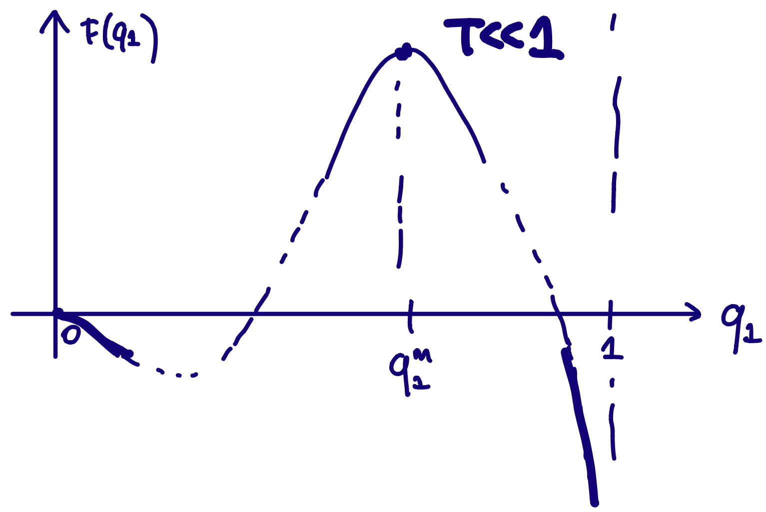
One sees that $\begin{cases} F(q_1 = 0) = 0 \\ F(q_1 = 1) = -\infty \end{cases}$

To vanish at some point $q_1 \neq 0$, the function $F(q_1)$ must be non-monotonic. Take $\boxed{p = 3}$.

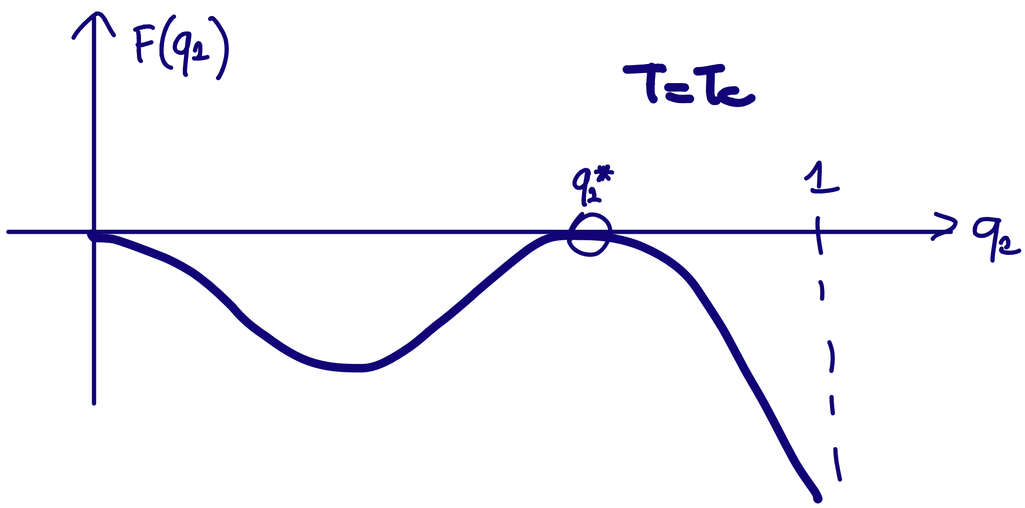
One can show that $F'(q_1^m) = 0$ for $q_1^m = \dfrac{1 + \sqrt{1 - \frac{4}{3}1/\beta^2}}{2}$

and $F(q_1^m) \overset{\beta \to \infty}{\simeq} \beta^2 + \log(1/\beta^2) \xrightarrow{\beta \to \infty} \infty$

Therefore, for small T (large β) the function $F(q_1)$ must cross zero at some $q_1 > 0$ because:



In fact, there exists a T_c such that:



Numerically, $\beta_c \simeq 1.2066 = 1/T_c$ for $p=3$.

Thus:

(a) $T > T_c$: $\mu^* = 1$, $q_0^* = 0 = q_1^*$. Paramagnet

(b) $T = T_c$: $\mu^* = 1$, $q_0^* = 0$, $q_1^* > 0$ – jump in q_1^*!

(c) $T < T_c$: $\mu^* < 1$, $q_0^* = 0$, $q_1^* > 0$: as $T \to 0$, $q_1^* \longrightarrow 1$ and $\mu^* \to 0$.
The overlap $q_1^* = q_{EA}$ changes with T in the low-T phase, at variance with REM

Extra: derivation 1RSB free-energy.

In Problems 3 we obtained:

$$\overline{Z^n} = \int \prod_{a<b} dq_{ab}\, e^{N A_n[Q] + o(N)}$$

$$A_n[Q] = \frac{\beta^2}{2} \sum_{a,b} q_{ab}^p + \frac{n}{2} \log(2\pi e) + \frac{1}{2} \log \det(Q)$$

We now plug the 1RSB structure of Q.

$$\sum_{a,b} q_{ab}^p = n + \frac{n}{\mu} \mu(\mu-1)\, q_1^p + \left(n(n-1) - \frac{n}{\mu} \mu(\mu-1) \right) q_0^p$$

The expression for the determinant can be obtained diagonalizing Q and taking the product of the eigenvalues with the correct degeneracy. This gives:

$$\log \det Q = n \frac{(\mu-1)}{\mu} \log(1-q_1) + \frac{n-\mu}{\mu} \log\left[\mu(q_1-q_0) + 1-q_1\right]$$
$$+ \log\left[n q_0 + \mu(q_1-q_0) + 1 - q_1\right]$$

Since we need $n \to 0$, we now expand $A_n[Q]$ around $n=0$ up to linear order. We get:

$$\sum_{a,b} q_{ab}^{p} = n\left[1 + (\mu-1)\, q_1^{p} - \mu\, q_0^{p}\right] + \mathcal{O}(n^2)$$

and

$$\log\left[n q_0 + \mu(q_1-q_0) + 1 - q_1\right] =$$

$$= \log\left[\left(\mu(q_1-q_0)+1-q_1\right)\cdot\left(1 + \frac{n q_0}{\mu(q_1-q_0)+1-q_1}\right)\right] =$$

$$= \log\left[\mu\,(q_1-q_0)+1-q_1\right] + \frac{n\, q_0}{\mu(q_1-q_0)+1-q_1} + \mathcal{O}(n^2)$$

Thus:

$$A_n[Q] = n\left\{\frac{\beta^2}{2}\left[1+(\mu-1)\,q_1^{p} - \mu\, q_0^{p}\right] + \frac{\log(2\pi e)}{2} + \frac{1}{2}\,\frac{\mu-1}{\mu}\log(1-q_1) + \frac{1}{2}\,\frac{1}{\mu}\log\left[\mu(q_1-q_0)+1-q_1\right] + \frac{1}{2}\,\frac{q_0}{\mu(q_1-q_0)+1-q_1}\right\} + \mathcal{O}(n^2)$$

$$= n\, A_0[Q] + \mathcal{O}(n^2)$$

Therefore, once the saddle-point is performed:

$$\overline{Z^n} = e^{N\{n A_0[Q^*] + \mathcal{O}(n^2)\} + o(N)}$$

$\Rightarrow f = \lim_{N\to\infty}\lim_{n\to 0} -\frac{1}{\beta}\left(\frac{\overline{Z^n}-1}{Nn}\right) = -\frac{1}{\beta} A_0[Q^*]$ which is the given expression.