Introduction to Random Matrices

1 Recap: Matrices with real spectrum

We define $X = [X_{ij}]$ as an $N \times N$ matrix, where the elements X_{ij} are real or complex numbers. The N eigenvalues $\{\lambda_1, \ldots, \lambda_N\}$ of X are generally complex. However, there are two classes of matrices whose eigenvalues are real:

- real symmetric matrices,
- complex Hermitian matrices.

1.1 Real Symmetric Matrices

A matrix X is real and symmetric if $X_{ij} = X_{ji}$, which can be written as $X^t = X$, where X^t is the transpose of X.

A real symmetric matrix can be diagonalized by an *orthogonal* matrix O, such that $O^{-1} = O^t$. This means $O^t O = I$, where I is the $N \times N$ identity matrix.

Example

Let us consider the matrix:

$$X = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}.$$

This matrix is symmetric, as $X^t = X$. To diagonalize X, we find its eigenvalues and eigenvectors.

The eigenvalues λ are found by solving the characteristic equation:

$$\det(X - \lambda I) = 0,$$

where I is the identity matrix. Substituting X:

$$\det \begin{bmatrix} 4-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} = 0.$$

Expanding the determinant:

$$(4-\lambda)(3-\lambda)-1=\lambda^2-7\lambda+11=0.$$

Solving this quadratic equation gives the eigenvalues:

$$\lambda_1=5, \quad \lambda_2=2.$$

Next, we find the eigenvectors. For $\lambda_1=5$, solve $(X-5I)v_1=0$. This gives $v_1=\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$ (normalized). For $\lambda_2=2$, solve $(X-2I)v_2=0$. This gives $v_2=\begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$ (normalized). The orthogonal matrix O formed by the normalized eigenvectors is:

$$O=\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}.$$

It correspond to a rotation of 45 degrees with

$$O=\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}.$$

It The diagonalized form of X is:

$$D=O^tXO=\begin{bmatrix}5 & 0\\ 0 & 2\end{bmatrix}.$$

1.2 Complex Hermitian Matrices

A matrix X is complex and Hermitian if $X_{ij}=X_{ji}^*$, which can be written as $X^\dagger=X$, where $X^\dagger$ is the conjugate transpose of X.

A complex Hermitian matrix can be diagonalized by a *unitary* matrix U, such that $U^{-1}=U^\dagger$. This means $U^\dagger U=I$.

Example

Consider the Hermitian matrix:

$$X=\begin{bmatrix}2 & i\\ -i & 3\end{bmatrix}.$$

This matrix satisfies $X^\dagger=X$. Its eigenvalues are $\lambda_1=1$ and $\lambda_2=4$, which are real.

The unitary matrix U that diagonalizes X is:

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix}.$$

The diagonalized form of X is:

$$D = U^{\dagger} X U = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$

2 Random Matrices Ensemble

We now study the case of random matrices by introducing a probability measure on the entries of the matrix X, i.e.,

$$P[\{X_{ij}\}]dX.$$

Here $dX = \prod dX_{ij}$ in the volume element. Depending on the symmetry of the matrix, we have a different number independent entries and then a different number of factors.

Exercise 1: the volume element

- For a real symmetric matrix, $dX = \prod_{1 \leq i \leq j \leq N} dX_{ij}$. How many factors does this product have?

- For a complex matrix, write $X_{ij} = x_{ij} + iy_{ij}$ with x_{ij}, y_{ij} real. For a complex Hermitian matrix, $dX = \prod_{1 \leq i \leq j \leq N} dX_{ij}$. How many factors does this product have?

- Consider real symmetric random matrices X and a generic orthogonal matrix O. The matrix $Y = O^T X O$ is also symmetric and real. Prove that the volume element is invariant under the orthogonal transformation, namely:

$$dX = \prod_i dX_{ii} \prod_{i<j} dX_{ij} = dY = \prod_i dY_{ii} \prod_{i<j} dY_{ij}.$$

Hint:

– *The* $\frac{N(N+1)}{2}$ *independent entries of the matrix* Y *can be written as a linear combination of the* $\frac{N(N+1)}{2}$ *independent entries of the matrix* X*:*

$$Y_{ij} = \sum_{kl} J_{ij,kl} X_{kl}.$$

Determine the size of the matrix J *and show that*

$$J_{ij,kl} = O^T_{ik} O_{jl}.$$

– *For a linear mapping, the transformation of the volume element is written as:*

$$dX = |\det(J)|\, dY.$$

Hence, to prove the theorem, it is sufficient to show that J *is orthogonal, namely:*

$$J^T J = \mathbb{I},$$

where $\mathbb{I}$ *is the identity matrix.*

This invariance is intuitive: orthogonal transformations are simple rotations of the coordinate system, and the volume element does not change under rotations. This invariance is also very important for rotationally invariant matrices, which we will study soon.

2.1 Wigner Random Matrices

A matrix is called a Wigner matrix if its independent entries (i.e., the entries not fixed by the symmetries) are independent random variables (i.e., there are no correlations between entries). For real symmetric matrices, a Wigner matrix ensemble is written as:

$$P[\{X_{ij}\}] = \prod_{i=1}^{N} f_i(X_{ii}) \prod_{1 \le i < j \le N} f_{ij}(X_{ij}).$$

2.2 Random Matrices Invariant Under Rotation

An ensemble of random matrices is said to be rotationally invariant if its probability distribution remains unchanged under orthogonal transformations (for real matrices) or unitary transformations (for complex matrices).

In the real case, for all matrices X in the ensemble and for any orthogonal transformation O, we have:

$$P[X] = P[O^T X O].$$

Key Properties:

- We showed that the volume element is invariant under rotation. Hence, if $Y = O^T X O$, we can write:

$$P[X]\,dX = P[Y]\,dY.$$

- The distribution of rotationally invariant random matrices does not depend on the choice of coordinate system. It is determined solely by the eigenvalues of the matrix. The eigenvectors are uniformly distributed on the unit sphere (of N dimensions for real matrices).

- Matrices from the Gaussian Orthogonal Ensemble (GOE) and Gaussian Unitary Ensemble (GUE) are classic examples of rotationally invariant matrices.

Exercise 2: GOE - The Distribution of the Entries

The Gaussian Orthogonal Ensemble (GOE) is defined by the following distribution:

$$P(X) \sim \exp\left(-a \operatorname{Tr} X^2\right),$$

where a is a positive constant and X is a real symmetric matrix.

- Show that the ensemble is rotationally invariant.

- Derive the probability distribution of the diagonal and off-diagonal entries of the matrix X with the correct normalization. Explain why the GOE is both rotationally invariant and a Wigner matrix.

It is possible to show that Gaussian rotationally invariant ensembles are the only ones that are also Wigner matrices.

Eigenvectors of the GOE

An eigenvector of a GOE matrix of size $N \times N$ is a point uniformly distributed on the hypersphere of dimension N. The simplest way to draw a point uniformly on the hypersphere is to generate N independent Gaussian random variables with zero mean and variance $1/N$. Indeed, as we have seen, the N-dimensional Gaussian distribution exhibits the correct spherical symmetry. For very large N, the variance of $1/N$ for each component ensures normalization. For moderate N, normalization must be enforced as an additional global constraint.

Exercise 3: GOE Eigenvectors in the Large N Limit

- Show that at large N, two independent vectors drawn from the unit hypersphere become orthogonal as $N \to \infty$.

2.3 Eigenvalues of GOE

Given a matrix X from the Gaussian Orthogonal Ensemble (GOE), we can diagonalize it as:

$$X = O^t \Lambda O,$$

where:

- $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$ is the diagonal matrix of eigenvalues,
- O is an orthogonal matrix ($O^t O = I$) containing the eigenvectors of X.

Volume Element Transformation

The volume element dX is expressed in terms of the eigenvalues $\{\lambda_i\}$ and the orthogonal matrix O, as:

$$dX = J(\{\lambda_i\})\, d\lambda_1\, d\lambda_2 \ldots d\lambda_N\, d\mu(O),$$

where:

- $J(\{\lambda_i\})$ is the Jacobian determinant of the transformation from X to (Λ, O). It is a function of the eigenvalues only.
- $d\mu(O)$ is volume element associated with the eigenvectors. It is a uniform measure of the angle of the hypersphere.

The Jacobian $J(\{\lambda_i\})$ is not simple to compute. For GOE is given by

$$J(\{\lambda_i\}) = \prod_{i<j} |\lambda_i - \lambda_j|.$$

The final form of the volume element is:

$$dX = \prod_{i<j} |\lambda_i - \lambda_j|^\beta\, d\lambda_1 d\lambda_2 \ldots d\lambda_N\, d\mu(O).$$

with $\beta = 1$ for GOE and $\beta = 2$ for GUE. We conclude that the joint distribution of GOE and GUE has the form

$$P(\{\lambda_i\}) \propto \prod_{i<j} |\lambda_i - \lambda_j|^\beta \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^N \lambda_i^2 \right).$$

Here and below we have $a = 1/(2\sigma^2)$.

Exercise 4: GOE for $N = 2$

We recover the previous result for the simple $N = 2$ case of the GOE (for simplicity we set $\sigma = 1$). Consider the 2×2 matrix:

$$X = \begin{bmatrix} x_1 & x_3 \\ x_3 & x_2 \end{bmatrix},$$

with eigenvalues λ_1 and λ_2, and the matrix of eigenvectors:

$$O = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Show that the Jacobian of the transformation is $|\lambda_1 - \lambda_2|$.
Hint:

- *Show that:*

$$x_1 = \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta,$$
$$x_2 = \lambda_2 \cos^2\theta + \lambda_1 \sin^2\theta,$$
$$x_3 = (\lambda_1 - \lambda_2)\cos\theta \sin\theta.$$

- *Compute the Jacobian:*

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial \lambda_1} & \frac{\partial x_2}{\partial \lambda_1} & \frac{\partial x_3}{\partial \lambda_1} \\ \frac{\partial x_1}{\partial \lambda_2} & \frac{\partial x_2}{\partial \lambda_2} & \frac{\partial x_3}{\partial \lambda_2} \\ \frac{\partial x_1}{\partial \theta} & \frac{\partial x_2}{\partial \theta} & \frac{\partial x_3}{\partial \theta} \end{vmatrix}.$$

After a little computation of the determinant of the 3×3 Jacobian matrix, you will find that:

$$J = |\lambda_1 - \lambda_2|.$$

Thus, you can conclude:

$$P(x_1, x_2, x_3) = \exp\left[-\frac{1}{2}(\lambda_1^2 + \lambda_2^2)\right] |\lambda_1 - \lambda_2| \, d\lambda_1 \, d\lambda_2 \, d\theta.$$

The Jacobian you just computed shows that, even though the entries of the GOE are independent, their eigenvalues are not. This is a very general property of the spectrum of correlated systems. It is called *level repulsion* because the eigenvalues repel each other.

Exercise 5: the Wigner surmise

We define $s = |\lambda_1 - \lambda_2|$ as the level spacing between the two eigenvalues. Compute the probability distribution $P(s)$.

You should find the celebrated Wigner surmise:

$$P(s) = \frac{s}{2} \exp\left(-\frac{s^2}{4}\right).$$

This distribution is exact for $N = 2$ but works remarkably well for any GOE, even though it is not exact.

3 Spectral properties in the large N limit

Semicircle Law

Both GOE and GUE matrices, when $N \to \infty$ follow the famous Wigner semicircle law for the density of eigenvalues, which is defined as the marginal density obtained from the joint distribution of all eigenvalues. In particular, for a GOE matrix X with diagonal elements having variance σ^2, the eigenvalue density is given by:

$$\rho(\lambda) = \frac{1}{2\pi N\sigma^2}\sqrt{4N\sigma^2 - \lambda^2},$$

for $|\lambda| \leq 2\sqrt{N}\sigma$, and $\rho(\lambda) = 0$ otherwise.

Tracy Widom distribution

Since the density in the large N limit is defined on a finite support $[-2\sqrt{N}\sigma, 2\sqrt{N}\sigma]$ which scales like $\sqrt{N}$, one could guess that the maximal (or minimal since the distribution is symmetric) eigenvalue will be given by the edge of the support. Indeed, at zero-th order $\lambda_{\max} = 2\sqrt{N}\sigma$. The corrections to the zero-th order are given by the now famous Tracy-Widom distribution. Specifically

$$\lambda_{\max} = 2\sqrt{N}\sigma + \sigma N^{-1/6}\chi_\beta \ ,$$

where χ_β is a $\mathcal{O}(1)$ random variable following the β-Tracy Widom distribution with β being Dyson's index, defined as $\beta = 1$ for GOE matrices and $\beta = 2$ for GUE matrices. No explicit formula is known for the Tracy-Widom distribution it is usually expresses in terms of Painlevé-Transcedents but for most pratical purposes its tails usually suffice:

$$f_\beta(x) \sim \begin{cases} e^{-\frac{\beta}{24}|x|^3} & \text{when} \quad x \to -\infty \\ e^{-\frac{2\beta}{3}|x|^{3/2}} & \text{when} \quad x \to +\infty \end{cases}$$

Notice that the distribution is very asymmetric, its right tail is much heavier than its left tail. This is a result of the logarithmic repulsion in the bulk of the semi-circle. In order to push the maximum leftwards the logarithmic repulsion force us to shift the entire gas left which is extremely costly. Hence it's much easier for the maximum to move rightwards where it only has to fight against the overall harmonic potential.

4 Dyson log-gas interpretation

The stochastic model describing the N-eigenvalues of a matrix from the GOE or GUE ensemble is sometimes referred to as the Dyson log-gas. Remember the joint distribution of the eigenvalues:

$$P(\{\lambda_i\}) \propto \prod_{i<j} |\lambda_i - \lambda_j|^\beta \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{N}\lambda_i^2\right) .$$

Notice that we can re-write it as

$$P(\{\lambda_i\}) = \frac{1}{Z}\exp\left(-\beta\left[\frac{1}{2\tilde{\sigma}^2}\sum_{i=1}^{N}\lambda_i^2 + \frac{1}{2}\sum_{i\neq j}\log|\lambda_i - \lambda_j|\right]\right) = \frac{1}{Z}\exp(-\beta E(\{\lambda_i\})) ,$$

where $\tilde{\sigma} = \sigma\sqrt{\beta}$. This form should remind you results from statistical physics...

Dyson tried to imagine what kind of gas could have an equilbrium distribution described by the above joint probability density function. He finally proved that the N-eigenvalues of a random matrix from the GOE or GUE ensembles are equivalent to a gas of N one-dimensional particles on a line, which undergo independent diffusive motions, repel each other with a logarithmic interaction and are confined around the origin with a harmonic trap.