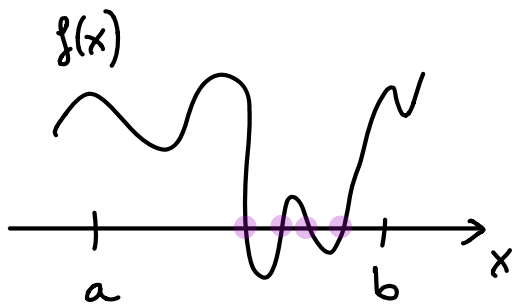


PROBLEM 5

Landscapes & Kac-Rice (1/2)

Problem 5: THE KR METHOD & THE COMPLEXITY

1 THE KAC-RICE FORMULA



One can write

$$N = \int_a^b dx \, \delta(x - f^{-1}(0))$$

Using the formula of change of variables of $\delta(\cdot)$,

$$\delta(y - F(x)) = \frac{1}{|F'(x)|} \delta(x - F^{-1}(y))$$

we have $N = \int_a^b dx \, \delta(f(x)) |f'(x)|$ and the

Jacobian

formula is obtained taking the average.

Now, we want to count the number of points $\vec{\sigma} \in S_N$

such that $f(x) \equiv \nabla_{\perp} E(\vec{\sigma}) = 0$. Since this function is a vector, its derivative has to be replaced by a matrix, the Hessian: the modulus is replaced by the absolute value of the determinant.

Therefore, the Kac-Rice formula counting the total number of stationary points is:

$$\overline{N} = \int_{S_N} d\vec{\sigma} \overline{\delta(\nabla_{\perp} E(\vec{\sigma})) |\det \nabla_{\perp}^2 E(\vec{\sigma})|}$$

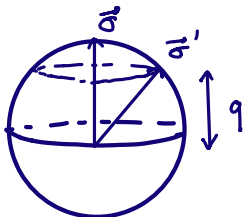
To count only those having a given value of energy density, one adds the constraint:

$$\delta(E(\vec{\sigma}) - N\epsilon).$$

[2] STATISTICAL ROTATIONAL INVARIANCE

The correlations read: $\overline{E(\vec{\sigma}) E(\vec{\sigma}')} = N \left(\frac{\vec{\sigma} \cdot \vec{\sigma}'}{N} \right)^p$

Given two configurations on the sphere, the correlation between the corresponding energies depends only on the overlap between the configurations



The overlap $q = \vec{\sigma} \cdot \vec{\sigma}' / N$ is rotational invariant, and so are the correlations.

The Kac-Rice formula reads:

$$\overline{N(\varepsilon)} = \int_{S_N} d\vec{\sigma} \, p_\varepsilon(\vec{\sigma})$$

The fact that the correlations, and thus the statistics of $E(\vec{\sigma})$, is rotational invariant implies that $p_\varepsilon(\vec{\sigma})$ does not depend on the particular configuration $\vec{\sigma}$, but it is the same for all $\vec{\sigma}$. We fix one value for the vector, $\vec{1} = (1, \dots, 1)$, and get:

$$\overline{N(\varepsilon)} = \int_{S_N} d\vec{\sigma} \, p_\varepsilon(\vec{\sigma}) = p_\varepsilon(\vec{1}) \int_{S_N} d\vec{\sigma} \simeq p_\varepsilon(\vec{\sigma}) \underbrace{(2\pi e)^{N/2}}_{\text{volume of sphere}}$$

[3] GAUSSIANTITY & CORRELATIONS

- The quantity $E(\vec{1}) = \sum_{i_1 < \dots < i_p} J_{i_1 \dots i_p}$ is a sum of independent Gaussian variables, and thus it is a gaussian variable with zero mean and:

$$\overline{(E(\vec{1}))^2} = N$$

Its distribution is $\frac{dE}{\sqrt{2\pi N}} e^{-E^2/2N}$; in terms of $\varepsilon = E/N$, this is $\sqrt{\frac{N}{2\pi}} d\varepsilon e^{-\frac{N\varepsilon^2}{2}}$

- The vectors $\nabla_{\perp} E(\vec{r})$ are obtained subtracting to $\nabla E(\vec{r})$ the component parallel to $\vec{r}$: one remains with $(N-1)$ components that are gaussian, with covariances:

$$\overline{(\nabla_{\perp} E)_{\alpha} (\nabla_{\perp} E)_{\beta}} = p \delta_{\alpha\beta} \quad \text{and with zero average.}$$

The probability density is:

$$\mathcal{P}(\nabla_{\perp} E) = \frac{1}{(2\pi p)^{\frac{N-1}{2}}} \prod_{\alpha=1}^{N-1} e^{-\frac{(\nabla_{\perp} E)_{\alpha}^2}{2p}}$$

$$\text{which computed at } \nabla_{\perp} E = 0 \text{ gives } \mathcal{P}(\vec{0}) = \left(\frac{1}{2\pi p} \right)^{\frac{N-1}{2}}$$

- Since $\nabla_{\perp} E(\vec{r})$ is independent of $E(\vec{r})$, $\nabla_{\parallel} E(\vec{r})$, we have

$$\overline{|\det \nabla_{\perp}^2 E(\vec{r})| \cdot \delta(\nabla_{\perp} E(\vec{r})) \cdot \delta(E(\vec{r}) - N\varepsilon)} =$$

$$= \overline{|\det \nabla_{\perp}^2 E(\vec{r})| \delta(E(\vec{r}) - N\varepsilon)} \cdot \overline{\delta(\nabla_{\perp} E(\vec{r}))}$$

$$= \overline{|\det \nabla_{\perp}^2 E(\vec{r})| \delta(E(\vec{r}) - N\varepsilon)} \cdot \mathcal{P}(\nabla_{\perp} E(\vec{r}) = 0)$$

$$= \overline{|\det \nabla_{\perp}^2 E(\vec{r})| \delta(E(\vec{r}) - N\varepsilon)} \cdot \left(\frac{1}{2\pi p} \right)^{\frac{N-1}{2}}$$

- Finally, the average of the determinant can be written as a conditional average, using the general relation:

$$\begin{aligned} \overline{X \cdot \delta(Y-y)} &= \int dx dy' \underbrace{P(x, y')}_{P(x|y)p(y)} \cdot x \cdot \delta(y' - y) = \left(\int dx P(x|y) \cdot x \right) \cdot p(y) \\ &= \overline{X|Y=y} P(Y=y) \\ &\quad \uparrow \text{conditional average} \end{aligned}$$

Therefore:

$$\overline{|\det \nabla_{\perp}^2 E(\vec{I})| \delta(E(\vec{I}) - N\varepsilon)} = \overline{|\det \nabla_{\perp}^2 E(\vec{I})| \mid E(\vec{I}) = N\varepsilon} \cdot P(E = N\varepsilon)$$

$\uparrow$ conditional average

Using the fact that the conditional law of the Hessian matrix $\nabla_{\perp}^2 E(\vec{I})$ is the same as matrices $(M - p\varepsilon \mathbb{1})$, where M is symmetric with Gaussian entries, we have that this term reduces to:

$$\overline{|\det \nabla_{\perp}^2 E(\vec{I})| \delta(E(\vec{I}) - N\varepsilon)} = \overline{|\det(M - p\varepsilon \mathbb{1})|} \cdot \sqrt{\frac{N}{2\pi}} e^{-\frac{N\varepsilon^2}{2}}$$

Combining everything:

$$\begin{aligned} \overline{N(\varepsilon)} &= (2\pi e)^{N/2} \left(\frac{1}{2\pi p} \right)^{\frac{N-1}{2}} \sqrt{\frac{N}{2\pi}} e^{-\frac{N\varepsilon^2}{2}} \overline{\left| \det(M - p\varepsilon \mathbb{1}) \right|}^M \\ &= e^{\frac{N}{2} \log\left(\frac{e}{p}\right) - \frac{N\varepsilon^2}{2} + \log \overline{\left| \det(M - p\varepsilon \mathbb{1}) \right|} + o(N)}, \end{aligned}$$

where the average of the determinant is over the ensemble of matrices M that are symmetric, and have independent random entries $M_{\alpha\beta}$ with $\alpha \leq \beta$, with gaussian statistics.

