

Exercises 7-8: Interfaces

Valentina Ros, Alberto Rosso

Exercise 7: Edwards–Wilkinson with stationary initial condition

Consider an Edwards–Wilkinson interface in $1 + 1$ dimensions, at temperature T , of length L with periodic boundary conditions:

$$\partial_t h(x, t) = \nu \partial_x^2 h(x, t) + \eta(x, t),$$

where $\eta(x, t)$ is a Gaussian white noise with zero mean and variance

$$\langle \eta(x, t) \eta(x', t') \rangle = 2T \delta(x - x') \delta(t - t').$$

The solution can be written in Fourier space:

$$\hat{h}_q(t) = \hat{h}_q(0) e^{-\nu q^2 t} + \int_0^t ds e^{-\nu q^2 (t-s)} \eta_q(s). \quad (1)$$

where we used the Fourier decomposition

$$\hat{h}_q(t) = \frac{1}{L} \int_0^L dx e^{iqx} h(x, t), \quad h(x, t) = \sum_q e^{-iqx} \hat{h}_q(t), \quad (2)$$

and the corresponding noise correlations

$$\langle \eta_{q_1}(t') \eta_{q_2}(t) \rangle = \frac{2T}{L} \delta_{q_1, -q_2} \delta(t - t') \quad (3)$$

with $q = 2\pi n/L$, $n = \dots, -1, 0, 1, \dots$

In class, we computed the mean square displacement of a point, starting from a flat interface at $t = 0$, i.e. $h(x, 0) = 0$. We found

$$\langle \Delta h_{\text{flat}}^2 \rangle = 2 \frac{T}{L} t + \begin{cases} T \sqrt{\frac{2t}{\pi\nu}}, & t \ll L^2, \\ \frac{T}{\nu} \frac{L}{12}, & t \gg L^2. \end{cases}$$

The first term describes the diffusion of the center of mass of the interface, while the second comes from the contribution of the non-zero Fourier modes and corresponds to the width of the interface.

Now, consider the case where the initial interface configuration $h(x, 0)$ is drawn from the equilibrium distribution at temperature T :

$$P_{\text{stat.}}[h] \propto \exp \left[-\frac{\nu}{2T} \int_0^L dx (\partial_x h)^2 \right]. \quad (4)$$

For simplicity we set the initial center of mass to zero, namely $\hat{h}_{q=0}(0) = 0$ for all realizations.

We consider the mean square displacement at $x = 0$. The average is now performed both over the thermal noise $\langle \cdot \rangle$ and over the initial condition $\overline{\cdot}$:

$$\overline{\langle \Delta h^2 \rangle} = \overline{\langle [h(0, t) - h(0, 0)]^2 \rangle} = \overline{\langle h^2(0, t) \rangle} + \overline{\langle h^2(0, 0) \rangle} - 2 \overline{\langle h(0, t) h(0, 0) \rangle}. \quad (5)$$

1. From the Gaussian ensemble of initial conditions, compute

$$\overline{\hat{h}_{q_1}(0) \hat{h}_{q_2}(0)}.$$

Hint. Write the integral in Eq. (4) using the Fourier modes and use $\int_0^L dx e^{iqx} = L \delta_{q,0}$.

2. Show that

$$\overline{\langle h^2(0,0) \rangle} = \frac{T}{\nu L} \sum_{q \neq 0} \frac{1}{q^2}, \quad \overline{\langle h(0,t)h(0,0) \rangle} = \frac{T}{\nu L} \sum_{q \neq 0} \frac{e^{-\nu q^2 t}}{q^2}.$$

3. Show that

$$\overline{\langle h^2(0,t) \rangle} = A(t) + \langle \Delta h_{\text{flat}}^2 \rangle,$$

where $A(t)$ depends solely on the initial condition (and not on the thermal noise), and prove that

$$A(t) = \frac{T}{\nu L} \sum_{q \neq 0} \frac{e^{-2\nu q^2 t}}{q^2}.$$

4. Hence you can write

$$C(t) \equiv \overline{\langle \Delta h^2 \rangle} - \langle \Delta h_{\text{flat}}^2 \rangle = \frac{2T}{\nu L} \sum_{n=1}^{\infty} \frac{(1 - e^{-\nu(\frac{2\pi n}{L})^2 t})^2}{(\frac{2\pi n}{L})^2}.$$

Estimate the value of $C(t)$ for $t \gg L^2$.

5. Estimate $C(t)$ for $t \ll L^2$ and large L . *Hint.* Write the series as an integral using the continuum variable $z = \frac{2\pi n}{L}$. It is helpful to know that

$$\int_0^{\infty} ds \frac{(1 - \exp(-s^2))^2}{s^2} = \sqrt{\pi} (2 - \sqrt{2}). \quad (6)$$

Provide the two asymptotic behaviors of $\overline{\langle \Delta h^2 \rangle}$.

Exercise 8: Larkin model (quenched random force) at $T = 0$

We consider a d -dimensional elastic interface $h(\vec{r}, t)$ embedded in a medium of dimension $D = d + 1$. Its energy functional is

$$E_{\text{pot}}[h] = \int d^d r \left[\frac{\nu}{2} (\nabla h)^2 + V(h(\vec{r}), \vec{r}) \right], \quad (7)$$

where the first term represents elasticity and V is a quenched random potential.

A. Larkin introduced a simplified model in which the disorder potential is linearized around a reference height:

$$V(h(\vec{r}), \vec{r}) \simeq V_0(\vec{r}) + F(\vec{r}) h(\vec{r}). \quad (8)$$

The resulting energy landscape is trivial (it is purely quadratic in h), but the model plays a key role in pinning theory.

Throughout the exercise, assume overdamped dynamics and set the mobility to 1.

(i) *Equation of motion.* Derive the equation of motion at zero temperature. *Tip: use* $\partial_t h(\vec{r}, t) = -\delta E_{\text{pot}} / \delta h(\vec{r}, t)$.

(ii) *Scaling analysis.* Assume that $F(\vec{r})$ is a quenched Gaussian random field with

$$\overline{F(\vec{r})} = 0, \quad \overline{F(\vec{r}) F(\vec{r}') } = D \delta^d(\vec{r} - \vec{r}'). \quad (9)$$

Predict the roughness exponent ζ_L and the dynamical exponent z_L . Determine the upper critical dimension d_{uc} above which the interface is flat. *Tip: in Fourier space, the stationary solution has the form $h_{\vec{q}} \propto F_{\vec{q}}/q^2$.*

(iii) *Center of mass.* For simplicity, consider a one-dimensional interface of size L with periodic boundary conditions, initially flat: $h(x, 0) = 0$. Let $h_0(t) \equiv \frac{1}{L} \int_0^L dx h(x, t)$ be the center of mass. Determine $\overline{h_0(t)}$ and $\overline{h_0(t)^2}$. *Tip: the $q = 0$ mode is driven by the spatial average of $F(x)$, which is time independent.*

(iv) *Structure factor.* Compute the structure factor and show:

$$\overline{\hat{h}_q(t)\hat{h}_{-q}(t)} = \frac{(1 - e^{-\nu q^2 t})^2}{\nu^2 q^4} \quad (10)$$

where $q = |\vec{q}|$. Find the relaxation time τ_q of the model and again determine ζ_L and z_L .

(iv) *Width of the interface.* Determine the explicit expression of the width of the interface at short and long times. Using

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \quad (11)$$

$$\int_0^{\infty} \frac{(1 - e^{-z^2})^2}{z^4} dz = \frac{4\sqrt{\pi}}{3}(\sqrt{2} - 1) \quad (12)$$