# SOLUTIONS TD3

## The replica method (1/2)

## Problem 3.1: Correlations, p-spin vs REM

### 1 Energy correlations

We have: $\overline{E(\vec{\sigma})E(\vec{\tau})} = \sum_{1\leq i_1<\dots<i_p\leq N} \sum_{1\leq j_1\dots\leq N} \overline{J_{i_1\dots i_p} J_{j_1\dots j_p}} \sigma_{i_1}\cdots\sigma_{i_p}\tau_{j_1}\cdots\tau_{j_p}$

The random couplings are independent, and therefore the average is non-zero only whenever all the indices are the same. Using the expression of the variance:

$$\overline{E(\vec{\sigma})E(\vec{\tau})} = \sum_{1\leq i_1<\dots<i_p\leq N} \frac{p!}{N^{p-1}} \sigma_{i_1}\tau_{i_1}\sigma_{i_2}\tau_{i_2}\cdots\sigma_{i_p}\tau_{i_p}$$

Now, the constraint on the non-repeating indices can be released using that:

$$\sum_{i_1<\dots<i_p} \simeq \frac{1}{p!}\sum_{i_1,\dots,i_p}$$

and thus $\overline{E(\vec{\sigma})E(\vec{\tau})} = N\left(\sum_{i_1=1}^{N} \frac{\sigma_{i_1}\tau_{i_1}}{N}\right)\cdots\left(\sum_{i_p=1}^{N} \frac{\sigma_{i_p}\tau_{i_p}}{N}\right) = N\,q^p$

where $q = q(\vec{\sigma},\vec{\tau})$ is the overlap.

One has $q \leq 1$ (Cauchy-Shwartz): therefore, when $p \to \infty$ $q^p \to 0$, and correlation vanish as in the REM.

# Problem 3.2: THE ANNEALED FREE-ENERGY

## 1 ENERGY CONTRIBUTION

The averaged partition function is:

$$\overline{Z} = \int_{S_N} d\vec{\sigma}\, \overline{e^{-\beta E(\vec{\sigma})}} = \int_{S_N} d\vec{\sigma}\, \overline{e^{-\beta \sum_{i_1<\dots<i_p} J_{i_1\dots i_p}\sigma_{i_1}\cdots\sigma_{i_p}}} \overset{\text{independence}}{=} \int_{S^N} d\vec{\sigma} \prod_{i_1<\dots<i_p} \overline{e^{-\beta J_{i_1\dots i_p}\sigma_{i_1}\dots\sigma_{i_p}}}$$

where $\int_{S_N} d\vec{\sigma}$ is the integral on the surface of the sphere in dimension N.

By definition, $f_a = \lim_{N\to\infty} \frac{1}{N} \log \overline{Z}$.

Performing the Gaussian integral (e.g. by completing the square) we get:

$$\overline{e^{-\beta J \sigma_{i_1} \dots \sigma_{i_p}}} = e^{\frac{\beta^2}{2} \sigma_{i_1}^2 \dots \sigma_{i_p}^2 \frac{p!}{N^{p-1}}} \quad \text{and thus:}$$

$$\overline{Z} = \int_{S_N} d\vec{\sigma} \; e^{\frac{\beta^2 N}{2} \left( p! \sum_{i_1 < \dots < i_p} \frac{\sigma_{i_1}^2}{N} \cdots \frac{\sigma_{i_p}^2}{N} \right)} = e^{\frac{\beta^2 N}{2}} \int_{S_N} d\vec{\sigma}.$$

Using that $\sum_i \sigma_i^2 = N$

## 2 ENTROPY CONTRIBUTION

The Stirling formula implies $\left(\frac{N}{2}\right)! \overset{N \gg 1}{\simeq} e^{-\frac{N}{2}} \left(\frac{N}{2}\right)^{N/2} = e^{-\frac{N}{2} + \frac{N}{2} \log\left(\frac{N}{2}\right)}$

and thus

$$\int_{S_N} d\vec{\sigma} = \frac{(\pi N)^{N/2}}{\left(\frac{N}{2}\right)!} \simeq e^{\frac{N}{2} \left[ \log(\pi N) + 1 - \log\left(\frac{N}{2}\right) \right] + o(N)}$$

$$= e^{\frac{N}{2} \log(2\pi e) + o(N)}$$

Putting everything together, we find:

$$\overline{Z} = \exp\left\{ N \left( \frac{\beta^2}{2} + \frac{1}{2} \log(2\pi e) \right) + o(N) \right\}$$

and therefore $$f_a = -\frac{1}{\beta}\left[\frac{\beta^2}{2} + \frac{1}{2}\log(2\pi e)\right]$$

The difference comes from the entropic contribution $\int_{S_N} d\vec{\sigma}$, and it is due to the fact that the phase space of the spherical model is different from that of the REM, where spins are discrete variables $\pm 1$.

# Problem 3.3: Quenched Free-Energy, Replicas

## $\boxed{1}$ STEP 1: FROM QUENCHED RANDOMNESS TO INTERACTIONS

The n-th power of the partition function is:

$$Z^n = \left(\prod_{a=1}^{n} \int_{S^N} d\vec{\sigma}^a\right) \exp\left[-\beta \sum_{i_1<\dots<i_p} J_{i_1\dots i_p}\left(\sigma^1_{i_1}\cdots\sigma^1_{i_p} + \sigma^2_{i_1}\cdots\sigma^2_{i_p} + \dots + \sigma^n_{i_1}\cdots\sigma^n_{i_p}\right)\right]$$

When averaging over the couplings $J_{i_1\dots i_p}$, we use again the properties of INDEPENDENCE and GAUSSIANITY and get:

$$\overline{Z^n} = \left(\prod_{a=1}^{n}\int_{S_N} d\vec{\sigma}^a\right) \prod_{i_1<\dots<i_p} e^{\frac{\beta^2}{2}\frac{p!}{N^{p-1}}\left(\sigma^1_{i_1}\dots\sigma^1_{i_p} + \sigma^2_{i_1}\dots\sigma^2_{i_p} + \dots + \sigma^n_{i_1}\dots\sigma^n_{i_p}\right)^2}$$

The square at the exponent can be re-written as:

$$\sum_{a=1}^{n} \sum_{b=1}^{n} (\sigma^a_{i_1} \sigma^b_{i_1}) \dots (\sigma^a_{i_p} \sigma^b_{i_p})$$

Therefore, using again that $\sum_{i_1 < \dots < i_p} \simeq \frac{1}{p!} \sum_{i_1, \dots, i_p}$

we obtain:

$$\overline{Z^n} = \left( \prod_{a=1}^{n} \int_{S_N} d\vec{\sigma}^a \right) e^{\frac{\beta^2}{2} N \sum_{a,b=1}^{n} \sum_{i_1, i_2, \dots i_p} \frac{(\sigma^a_{i_1} \sigma^b_{i_1})}{N} \cdots \frac{(\sigma^a_{i_p} \sigma^b_{i_p})}{N}}$$

$$= \left( \prod_{a=1}^{n} \int_{S^N} d\vec{\sigma}^a \right) e^{\frac{\beta^2}{2} N \sum_{a,b=1}^{n} \left( \frac{\vec{\sigma}^a \cdot \vec{\sigma}^b}{N} \right)^p} \qquad (*)$$

In this expression, the quenched randomness has disappeared, but the replicas are coupled.

> Step 1: start from expression with replicas decoupled, subject to same disorder.
> After averaging, end up with COUPLED REPLICAS (interacting theory), no disorder.

## 2 STEP 2: EMERGING ORDER PARAMETERS

The final expression of $\overline{Z^n}$ shows that the integrand depends on the variables $\sigma_i^a$ only through global quantities, the scalar products between the $\vec{\sigma}^a$.

We can therefore identify a set of functions, the overlaps between the replicas:

$$q^{ab} = q(\vec{\sigma}^a, \vec{\sigma}^b) = \sum_{i=1}^{N} \frac{\sigma_i^a \sigma_i^b}{N},$$

that are ORDER PARAMETERS of the theory, like the magnetization $m = \frac{1}{N}\sum_{i=1}^{N} \sigma_i$ in the mean-field Ising model. In particular, in (*) we can replace the integral over all possible configurations of the $\vec{\sigma}^a$ with an integral over all possible values of the overlaps, using:

functions

$$\int \prod_{a<b} dq_{ab}\, \delta\left(q(\vec{\sigma}^a, \vec{\sigma}^b) - q_{ab}\right) = 1$$

integration variables, numbers

Plugging this in (*) we obtain:

$$\left(\prod_{a=1}^{n}\int_{S_N} d\vec{\sigma}^a\right) e^{\frac{\beta^2}{2}N\sum_{a,b=1}^{n}\left(\frac{\vec{\sigma}^a\cdot\vec{\sigma}^b}{N}\right)^p}$$

$$=\left(\prod_{a=1}^{n}\int_{S_N} d\vec{\sigma}^a\right)\left(\int\prod_{a<b} dq_{ab}\,\delta\left(q(\vec{\sigma}^a,\vec{\sigma}^b)-q_{ab}\right) e^{\frac{\beta^2}{2}N\sum_{a,b=1}^{n}\left(\frac{\vec{\sigma}^a\cdot\vec{\sigma}^b}{N}\right)^p}\right)$$

exchange

$$=\int\prod_{a<b} dq_{ab}\left[\left(\prod_{a=1}^{n}\int_{S^N} d\vec{\sigma}^a\right)\prod_{a<b}\delta\left(q(\vec{\sigma}^a,\vec{\sigma}^b)-q_{ab}\right)\right] e^{\frac{\beta^2}{2}N\sum_{a,b=1}^{n}q_{ab}^p}$$

We call

$$V\left(\{q_{ab}\}_{a<b}\right)=\left(\prod_{a=1}^{n}\int_{S^N} d\vec{\sigma}^a\right)\prod_{a<b}\delta\left(q(\vec{\sigma}^a,\vec{\sigma}^b)-q_{ab}\right)=e^{N S[\{q_{ab}\}]+o(N)}$$

where $S[\ ]$ is the entropy of configurations satisfying the constraint on the overlaps being equal to $q_{ab}$.

We introduce the $n\times n$ matrix with components:

$$Q_{ab}=\begin{cases} q_{ab} & a<b \\ 1 & a=b \\ q_{ba} & b<a \end{cases}$$

OVERLAP MATRIX

Then it can be shown (exercise! solution below) that

$$V[Q] = e^{N S[Q] + o(N)}, \quad S[Q] = \frac{n}{2} \log(2\pi e) + \frac{1}{2} \log \det[Q]$$

and thus:

$$\overline{Z^n} = \int \prod_{a<b} dq_{ab} \, e^{N\left\{ \frac{\beta^2}{2} \sum_{a,b} q_{ab}^p + \frac{n}{2} \log(2\pi e) + \frac{1}{2} \log \det[Q] \right\}}$$

This theory now is expressed only in terms of Q:

$$\overline{Z^n} = \int \prod_{a<b} dq_{ab} \, e^{N A[Q] + o(N)}$$

$$A[Q] = \frac{\beta^2}{2} \sum_{a,b} q_{ab}^p + \frac{n}{2} \log(2\pi e) + \frac{1}{2} \log \det[Q]$$

Step 2: re-write the integral over configurations as integral over emerging order parameter $q_{ab}$. Same as magnetization for mean-field Ising:

$$Z_{Ising} = \sum_{\vec{S}=1}^{2^N} e^{-\frac{\beta J}{N} \sum_{\ell,m} S_\ell S_m} \xrightarrow[m[\vec{S}] = \frac{1}{N}\sum_i S_i]{} \sum_m \underbrace{N(m)}_{\substack{\text{number of configurations} \\ \text{with magnetization } m}} e^{-\beta J N m^2}$$

In the replica calculation, have $\frac{n(n-1)}{2}$ order parameters to integrate over. We started with $Nn$ variable: HUGE DIMENSIONALITY REDUCTION due to mean-field.

## [3] STEP 3: SADDLE-POINT, SELECTING THE TYPICAL

For large $N$, the integral over the space of $n \times n$ matrices $Q$ can be computed with a saddle-point approximation.

The derivative with respect to a matrix $Q$ has to be intended as the derivative w.r.t. its components:

$$\frac{\partial \sum_{c,d} q_{cd}^{p}}{\partial q_{ab}} = p\, q_{ab}^{p-1} \quad , \quad \frac{\partial}{\partial q_{ab}} \log \det [Q] = (Q^{-1})_{ab}$$

Therefore the saddle point equation reads:

$$\left. \frac{\beta^2}{2} p\, q_{ab}^{p-1} + \frac{1}{2} (Q^{-1})_{ab} \right|_{Q=Q^*} = 0 \qquad (a \neq b,\ n \to 0)$$

$Q^*$ = saddle-point value

To proceed, need to make assumptions on structure of $q_{ab}$ at the saddle point: a "VARIATONAL ANSATZ".

# Extra: Volume Term

$$V = \left( \prod_{a=1}^{n} \int d\vec{\sigma}^a \right) \prod_{a<b} \delta\left(q(\vec{\sigma}^a, \vec{\sigma}^b) - q_{ab}\right) \prod_a \delta\left(q(\vec{\sigma}^a, \vec{\sigma}^a) - 1\right)$$

$$= N^{\frac{n(n-1)}{2}+n} \left( \prod_{a=1}^{n} \int d\vec{\sigma}^a \right) \prod_{a<b} \delta\left(\vec{\sigma}^a \cdot \vec{\sigma}^b - N q_{ab}\right) \prod_a \delta\left(\vec{\sigma}^a \cdot \vec{\sigma}^a - N\right)$$

$$= N^{\frac{n(n+1)}{2}} \left( \prod_{a=1}^{n} \int d\vec{\sigma}^a \right) \int \prod_{a\leq b} \frac{d\lambda_{ab}}{\sqrt{2\pi}} \, e^{i \sum_{a\leq b} \lambda_{ab} \left(\vec{\sigma}^a \cdot \vec{\sigma}^b - Q_{ab} N\right)}$$

where $Q_{ab} = \begin{cases} q_{ab} & \text{if } a<b \\ 1 & \text{if } a=b \\ q_{ab} & \text{if } b<a \end{cases}$

$$= \left( \frac{N}{\sqrt{2\pi}} \right)^{\frac{n(n+1)}{2}} \int \prod_{a\leq b} d\lambda_{ab} \, e^{-iN \sum_{a\leq b} \lambda_{ab} Q_{ab}} \left[ \int \prod_{a=1}^{n} d\vec{\sigma}^a \, e^{i \sum_{a\leq b} \lambda_{ab} \vec{\sigma}^a \cdot \vec{\sigma}^b} \right]$$

Now, $\sum_{a\leq b} = \frac{1}{2} \sum_{a\neq b} (\cdot\cdot) + \sum_a (\cdot\cdot)$

Call $\tilde{\lambda}_{ab} = \begin{cases} -i \frac{\lambda_{ab}}{2} & a\neq b \\ -i \lambda_{aa} & a=b \end{cases}$

And get:

$$=\left(\frac{N}{\sqrt{2\pi}}\right)^{\frac{n(n+1)}{2}} \int \prod_{a\le b} d\lambda_{ab}\, e^{N\sum_{a,b}\tilde{\lambda}_{ab} Q_{ab}} \left[\int \prod_{a=1}^{n} d\vec{\sigma}^{a}\, e^{-\frac{1}{2}\sum_{a,b} 2\tilde{\lambda}_{ab}\vec{\sigma}^{a}\cdot\vec{\sigma}^{b}}\right]$$

Now, the integral over the variables $\sigma_i^a$ is a multivariate Gaussian integral. One has:

$$I = \int \prod_{a=1}^{n}\prod_{i=1}^{N} d\sigma_i^a \quad e^{-\frac{1}{2}\sum_{a,b}\sum_{ij}\sigma_i^a\left(2\tilde{\lambda}_{ab}\,\delta_{ij}\right)\sigma_j^b}$$

$$= (2\pi)^{\frac{Nn}{2}}\left[\det\left(2\tilde{\Lambda}\right)\right]^{-\frac{N}{2}} = e^{\frac{Nn}{2}\log(2\pi) - \frac{N}{2}\log\left(\det[2\tilde{\Lambda}]\right)}$$

Now, we are left with:

$$V = \left(\frac{N}{\sqrt{2\pi}}\right)^{\frac{n(n+1)}{2}} (2\pi)^{\frac{Nn}{2}} \int \prod_{a\le b} d\lambda_{ab}\, e^{N\sum_{a,b}\tilde{\lambda}_{ab}Q_{ab} - \frac{N}{2}\log\det[2\tilde{\Lambda}]}$$

$$= C_{N,n}\,(2\pi)^{\frac{Nn}{2}} \int \prod_{a\le b} d\tilde{\lambda}_{ab}\, e^{N\,\mathrm{tr}(\tilde{\Lambda}Q) - \frac{N}{2}\log\det[2\tilde{\Lambda}]}$$

This is an integral in matrix space that can be performed with a saddle point, which gives:

$$Q - \frac{1}{2}(\tilde{\Lambda})^{-1} = 0 \Rightarrow \tilde{\Lambda}^{*} = (2Q)^{-1}$$

$$\det[2\tilde{\Lambda}]\Big|_{\tilde{\Lambda}=\Lambda^{*}} \Rightarrow \det[Q^{-2}] = (\det Q)^{-2}$$

Putting everything together:

$$V = e^{\frac{Nn}{2}\log(2\pi) + \frac{Nn}{2} + \frac{N}{2}\log\det[Q]}$$

$$\overset{!}{=} e^{\frac{Nn}{2}\log(2\pi e) + \frac{N}{2}\log\det[Q]}$$

One can show that with this choice, the replica calculation reproduces the annealed calculation we did in Problem 1 $\longrightarrow$ EXERCISE

However, this is wrong in the low-T phase! There, fluctuations dominate and have to be captured by another structure of the matrix: 1RSB.

Assumption 2:
1-step RSB

$$(q_{ab})_{ab} = \begin{pmatrix} \overset{\longleftarrow m \longrightarrow}{\boxed{\begin{matrix} 1 & & q \\ & \ddots & \\ q & & 1 \end{matrix}}} & & \\ & \boxed{\begin{matrix} 1 & & q \\ & \ddots & \\ q & & 1 \end{matrix}} & \\ & & \boxed{\begin{matrix} 1 & & q \\ & \ddots & \\ q & & 1 \end{matrix}} \end{pmatrix}$$

meaning: assume that $n$ replicas organize in $n/m$ groups of size $m$, with the $m$ falling in similar configurat. at overlap $q$.

Notice: $m$ is arbitrary parameter, to be optimized-over in saddle point.