Problem 3: the replica calculation for p-spin

1 STEP 1: FROM QUENCHED RANDOMNESS TO INTERACTIONS

The n-th power of the partition function is:

$$Z^n = \left(\prod_{a=1}^{n} \int_{S^N} d\vec{\sigma}^a \right) \exp\left[-\beta \sum_{i_1 < .. < i_p} J_{i_1 .. i_p} \left(\sigma^1_{i_1} \cdots \sigma^1_{i_p} + \sigma^2_{i_1} \cdots \sigma^2_{i_p} + \cdots + \sigma^n_{i_1} \cdots \sigma^n_{i_p} \right) \right]$$

When averaging over the couplings $J_{i_1 .. i_p}$, we use again the properties of INDEPENDENCE and GAUSSIANITY and get:

$$\overline{Z^n} = \left(\prod_{a=1}^{n} \int_{S_N} d\vec{\sigma}^a \right) \prod_{i_1 < .. < i_p} e^{\frac{\beta^2}{2} \frac{p!}{N^{p-1}} \left(\sigma^1_{i_1} \cdots \sigma^1_{i_p} + \sigma^2_{i_1} \cdots \sigma^2_{i_p} + \cdots + \sigma^n_{i_1} \cdots \sigma^n_{i_p} \right)^2}$$

The square at the exponent can be re-written as:

$$\sum_{a=1}^{n} \sum_{b=1}^{n} \left(\sigma^a_{i_1} \sigma^b_{i_1} \right) \cdots \left(\sigma^a_{i_p} \sigma^b_{i_p} \right)$$

Therefore, using again that $\sum_{i_1 < .. < i_p} \simeq \frac{1}{p!} \sum_{i_1, .., i_p}$

we obtain:

$$\overline{Z^n} = \left(\prod_{a=1}^{n} \int_{S_N} d\vec{\sigma}^a\right) e^{\frac{\beta^2}{2} N \sum_{a,b=1}^{n} \sum_{i_1, i_2, \dots i_p} \left(\frac{\sigma^a_{i_1}\sigma^b_{i_1}}{N}\right) \cdots \left(\frac{\sigma^a_{i_p}\sigma^b_{i_p}}{N}\right)}$$

$$= \left(\prod_{a=1}^{n} \int_{S^N} d\vec{\sigma}^a\right) e^{\frac{\beta^2}{2} N \sum_{a,b=1}^{n} \left(\frac{\vec{\sigma}^a \cdot \vec{\sigma}^b}{N}\right)^p} \qquad (*)$$

In this expression, the quenched randomness has disappeared, but the replicas are coupled!

Step 1: start from expression with replicas decoupled, subject to same disorder. After averaging, end up with COUPLED REPLICAS (interacting theory), no disorder.

2 STEP 2: DIMENSIONALITY REDUCTION

The final expression of $\overline{Z^n}$ shows that the integrand depends on the variables σ_i^a only through global quantities, the scalar products between the $\vec{\sigma}^a$.

We can therefore identify a set of functions, the overlaps between the replicas:

$$q^{ab} = q(\vec{\sigma}^a, \vec{\sigma}^b) = \sum_{i=1}^{N} \frac{\sigma_i^a \sigma_i^b}{N},$$

that are ORDER PARAMETERS of the theory, like the magnetization $m = \frac{1}{N}\sum_{i=1}^{N} \sigma_i$ in the mean-field Ising model. In particular, in (*) we can replace the integral over all possible configurations of the $\vec{\sigma}^a$ with an integral over all possible values of the overlaps, using:

$$\int \prod_{a<b} \underset{\text{integration variables, numbers}}{dq_{ab}}\, \delta\Big(\overset{\text{functions}}{q(\vec{\sigma}^a, \vec{\sigma}^b)} - q_{ab}\Big) = 1$$

Plugging this in (*) we obtain:

$$\left(\prod_{a=1}^{n} \int_{S_N} d\vec{\sigma}^a\right) e^{\frac{\beta^2}{2} N \sum_{a,b=1}^{n} \left(\frac{\vec{\sigma}^a \cdot \vec{\sigma}^b}{N}\right)^p}$$

$$= \left(\prod_{a=1}^{n} \int_{S_N} d\vec{\sigma}^a\right)\left(\int \prod_{a<b} dq_{ab}\, \delta\big(q(\vec{\sigma}^a, \vec{\sigma}^b) - q_{ab}\big)\, e^{\frac{\beta^2}{2} N \sum_{a,b=1}^{n} \left(\frac{\vec{\sigma}^a \cdot \vec{\sigma}^b}{N}\right)^p}\right.$$

exchange

$$= \int \prod_{a<b} dq_{ab} \left[\left(\prod_{a=1}^{n} \int_{S^N} d\vec{\sigma}^a \right) \prod_{a<b} \delta\left(q(\vec{\sigma}^a, \vec{\sigma}^b) - q_{ab}\right) \right] e^{\frac{\beta^2}{2} N \sum_{a,b=1}^{n} q_{ab}^p}$$

We call the "volume" term:

$$V(\{q_{ab}\}_{a<b}) = \left(\prod_{a=1}^{n} \int_{S^N} d\vec{\sigma}^a \right) \prod_{a<b} \delta\left(q(\vec{\sigma}^a, \vec{\sigma}^b) - q_{ab}\right) = e^{N \tilde{S}[\{q_{ab}\}] + o(N)}$$

where $\tilde{S}[\,]$ is the entropy of configurations satisfying the constraint on the overlaps being equal to q_{ab}.

We introduce the $n \times n$ matrix with components:

$$Q_{ab} = \begin{cases} q_{ab} & a<b \\ 1 & a=b \\ q_{ba} & b<a \end{cases}$$

OVERLAP MATRIX

Then it can be shown (exercise! solution below) that

$$V[Q] = e^{N \tilde{S}[Q] + o(N)}, \quad \tilde{S}[Q] = \frac{n}{2} \log(2\pi e) + \frac{1}{2} \log \det[Q]$$

and thus:

$$\overline{Z^n} = \int \prod_{a<b} dq_{ab} \, e^{N \left\{ \frac{\beta^2}{2} \sum_{a,b} q_{ab}^p + \frac{n}{2} \log(2\pi e) + \frac{1}{2} \log \det[Q] \right\}}$$

This theory now is expressed only in terms of Q:

$$\overline{Z^n} = \int \prod_{a<b} dq_{ab} \, e^{N \mathcal{A}_n[Q] + o(N)}$$

" the action "

$$\mathcal{A}_n[Q] = \frac{\beta^2}{2} \sum_{a,b} q_{ab}^p + \frac{n}{2} \log(2\pi e) + \frac{1}{2} \log \det [Q]$$

Step 2: re-write the integral over configurations as integral over emerging order parameter q_{ab}. Same as magnetization for mean-field Ising:

$$Z_{Ising} = \sum_{\vec{S}=1}^{2^N} e^{-\frac{\beta J}{N} \sum_{e,m} S_e S_m} \xrightarrow[m[\vec{S}] = \frac{1}{N}\sum_i S_i]{} \sum_m \underbrace{\mathcal{N}(m)}_{\substack{\text{number of configurations} \\ \text{with magnetization } m}} e^{-\beta J N m^2}$$

In the replica calculation, have $\frac{n(n-1)}{2}$ order parameters to integrate over. We started with Nn variables: HUGE <u>DIMENSIONALITY REDUCTION</u> due to mean-field.

[3] STEP 3: SADDLE-POINT, SELECTING THE TYPICAL

For large N, the integral over the space of $n \times n$ matrices Q can be computed with a saddle-point approximation.

The derivative with respect to a matrix Q has to be intended as the derivative w.r.t. its components:

$$\frac{\partial \sum_{c,d} q_{cd}^{p}}{\partial q_{ab}} = \underbrace{2p}_{\text{symmetry } Q}\, q_{ab}^{p-1} \quad , \quad \frac{\partial}{\partial q_{ab}} \log \det[Q] = 2 (Q^{-1})_{ab}$$

Therefore the saddle point equations read

$$\beta^2 p\, q_{ab}^{p-1} + (Q^{-1})_{ab} \Big|_{Q=Q^*} = 0 \quad \text{for all } a<b \qquad (**)$$

Q^* = saddle-point value

To proceed, need to make assumptions on structure of q_{ab} at the saddle point: a "VARIATONAL ANSATZ".

Comment: there is one simple solution to the saddle point equations:

$$Q^*_a = \overbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{n \times n} = \overbrace{\mathbb{1}}^{n \times n}$$

This solves trivially the equations since both terms in (**) are zero.

In this case:

$$\overline{Z^n} = e^{N A_n[Q_n^*] + o(N)} = e^{Nn\left\{\frac{\log(2\pi e)}{2} + \beta^2/2\right\} + o(N)}$$

Using the replica trick:

$$f = -\frac{1}{\beta} \lim_{N\to\infty} \lim_{n\to 0} \frac{\overline{Z^n} - 1}{Nn} = -\frac{1}{\beta}\left\{\frac{\log(2\pi e)}{2} + \beta^2/2\right\}$$

Same result as the annealed.

At high T, this is the good solution.
There is a critical temperature T_c there is another solution, with lower free energy than the above!