

Problem 1: the energy landscape of the REM

1 ANNEALED ENTROPY

We write: $N_N(E)dE = \sum_{\alpha=1}^{2^N} \chi_{\alpha}(E) dE$ $\chi_{\alpha}(E) = \begin{cases} 1 & E_{\alpha} \in [E, E+dE] \\ 0 & \text{otherwise} \end{cases}$

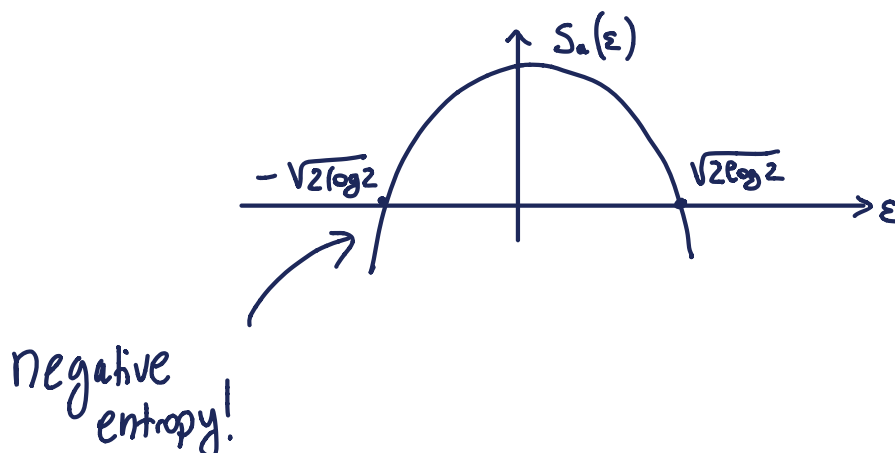
Taking the average:

$$\begin{aligned} \overline{N_N(E)} dE &= \sum_{\alpha=1}^{2^N} \overline{\chi_{\alpha}(E)} dE = \sum_{\alpha=1}^{2^N} P(E_{\alpha} \in [E, E+dE]) = \\ &= \sum_{\alpha=1}^{2^N} P(E) dE = 2^N \frac{1}{\sqrt{2\pi N}} e^{-E^2/2N} dE \\ &= e^{N(\log 2 - \frac{1}{2}(\frac{E}{N})^2) + o(N)} dE \end{aligned}$$

Introducing the energy density $\varepsilon = E/N$ we have

$$S_a(\varepsilon) = \lim_{N \rightarrow \infty} \frac{1}{N} \log \overline{N_N(E)} = \log 2 - \varepsilon^2/2.$$

This function is plotted below.



2 SELF-AVERAGING

Let us compute the second moment, as above:

$$\begin{aligned}
 \overline{(N_N(E)dE)^2} &= \sum_{\alpha=1}^{2^N} \sum_{\beta=1}^{2^N} \overline{\chi_{\alpha}(E) \chi_{\beta}(E) dE^2} = \sum_{\substack{\alpha, \beta=1 \\ (\alpha \neq \beta)}}^{2^N} \overline{\chi_{\alpha}(E) dE} \overline{\chi_{\beta}(E) dE} + \\
 &\quad \text{independence} \\
 &\quad + \sum_{\alpha=1}^{2^N} \overline{\chi_{\alpha}^2(E) dE^2} = \sum_{\alpha=1}^{2^N} \overline{\chi_{\alpha}(E) dE} \left(\sum_{\beta: \beta \neq \alpha} \overline{\chi_{\beta}} + 1 \right) dE \\
 &= \overline{N_N(E) dE} \left[(2^N - 1) \frac{e^{-E^2/2N}}{\sqrt{2\pi N}} + 1 \right] dE \\
 &= \overline{N_N(E) dE} \left[\overline{N_N(E) dE} + \left(1 - \frac{e^{-E^2/2N}}{\sqrt{2\pi N}} \right) dE \right]
 \end{aligned}$$

Therefore:

$$\frac{\overline{(N_N(E)dE)^2}}{(\overline{N_N(E)dE})^2} = 1 + \frac{1}{\overline{N_N(E)dE}} \left(1 - \frac{e^{-E^2/2N}}{\sqrt{2\pi N}} \right) dE$$

And $\overline{N_N(E)} = e^{NS_{\alpha}(E/N) + o(N)}$

When $S_{\alpha} > 0$, $\overline{N_N(E)}$ grows exponentially and

$$\lim_{N \rightarrow \infty} \frac{\overline{N_N^2(E)}}{(\overline{N_N(E)})^2} = 1 \Rightarrow N_N(E) \text{ is self-averaging,}$$

thus $S_{\alpha}(E) = S(E)$
for $|E| \leq \sqrt{2 \log 2}$

For $|\epsilon| > \sqrt{2\epsilon \log 2}$ the average decays to zero faster than the standard deviation: the fluctuations are not negligible and the large- N behavior is not controlled by the average.

The quantity is not self-averaging: sample-to-sample fluctuations will matter when N is large.

③ RARE EVENTS

Let us try to bound the probability to have configurations with $|\epsilon| > \sqrt{2\epsilon \log 2}$. It holds:

$$\begin{aligned}
 & \mathbb{P} \left(\begin{array}{l} \text{exists at least one configuration} \\ E_x \text{ such that } E_x \in [E, E+dE] \\ \text{with } E < -N\sqrt{2\epsilon \log 2} \end{array} \right) = \\
 &= \sum_{n=1}^{2^N} \mathbb{P} \left(\begin{array}{l} \text{exist } n \text{ configurations} \\ \text{in } [E, E+dE] \end{array} \right) \leq \sum_{n=1}^{2^N} n \mathbb{P} \left(\begin{array}{l} \text{exist } n \\ \text{configs in } [E, E+dE] \end{array} \right) \\
 &= \overline{N_n(E)} dE \Rightarrow \text{exponentially small.}
 \end{aligned}$$

Notice: this bound is a special case of Markov's inequality, $\mathbb{P}(X \geq a) \leq E[X]/a$.

applied to the random variable $X = \overline{N_n(E)} dE$ and $a=1$.

Since the probability to find a configuration is exponentially small in this region, the typical number of configurations is zero: $N^{\text{typ}}(\varepsilon)$ vanishes, and so $S(\varepsilon) = -\infty$.

Thus, putting everything together we have:

$$S(\varepsilon) = \begin{cases} S_{\text{alt}}(\varepsilon) = \log 2 - \varepsilon^2/2 & |\varepsilon| \leq \sqrt{2 \log 2} \\ -\infty & |\varepsilon| > \sqrt{2 \log 2} \end{cases}$$

4 ENERGY LANDSCAPE

At $T = \infty$, the Boltzmann measure is a flat measure over configurations. Most configurations have $\varepsilon = 0$, so this will be the equilibrium energy density.

The point where $S(\varepsilon) = 0$ gives the minimal (maximal) energy density at which one finds configurations with a probability that is $\Theta(1)$ as $N \rightarrow \infty$: it defines the TYPICAL VALUE of the ground state energy density. (i.e., equilibrium at $T = 0$)

This is consistent with what found in Lecture 1 by Alberto on extreme values:

$$E_{\min}^{\text{typ}} = a_M + b_M Z \quad a_M = E_{\min}^{\text{typ}} = -N \sqrt{2 \log 2}$$

