# Second Homework

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This is the second part of the Homework, that follows the one on Random Matrix Theory. The homework should be hand in by **Monday, March 24th 2025**.

## Problem 1. A generalized Random Energy Model (8 points)

In this problem we study a generalization of the Random Energy Model (REM). We assume that the system is made of $M = 2^N$ configurations with random independent energies $E_i$ distributed according to:

$$P_\delta(E) = C\, e^{-A(N)|E|^\delta}, \qquad C,\ A(N),\ \delta > 0. \tag{1}$$

We denote with $Z$ the partition function of the model, and set $\beta = 1/T$.

1. *Ground state and scaling [2 pt].* Using extreme value statistics, determine the typical value of the Ground State energy $E_{GS}^{\text{typ}}$. Set $A(N) = A \cdot N^\alpha$: which choice of $\alpha$ guarantees that the ground state is an extensive quantity? Recover the results obtained in the case of the REM discussed in class.

   Consider from now on $\delta > 1$, and $A(N) = A \cdot N^\alpha$ with the exponent $\alpha$ determined in point 1.

2. *Typical energy [2 pt].* Using a saddle point approximation, compute the function $g_\delta(\lambda)$ defined as:

$$\int_{-\infty}^{\infty} dE\, P_\delta(E)\, e^{-\lambda E} \equiv e^{N g_\delta(\lambda) + o(N)} \qquad \lambda > 0. \tag{2}$$

   *Hint:* assume that the saddle point value of the energy lies in the region $E \leq 0$.

3. *Annealed calculation [2 pt].* Setting $\lambda \to \beta$, compute the average partition function $\overline{Z}$ and the annealed free-energy density

$$f_a(T) = -T \lim_{N\to\infty} \frac{\log \overline{Z}}{N}. \tag{3}$$

   Show that the annealed entropy density $s_a(T) \equiv -\frac{df_a(T)}{dT}$ equals to:

$$s_a(T) = \log 2 - A[\delta\, A\, T]^{-\frac{\delta}{\delta-1}}. \tag{4}$$

   Determine the freezing temperature $T_f$ of the model for general $\delta$ (Hint: use that when $T = T_f$, the entropy vanishes). What do you expect to be the behavior of the free energy density obtained within the quenched formalism for both $T > T_f$ and $T < T_f$?

4. *Quenched calculation [2 pt].* Within a replica calculation with a 1RSB ansatz, one can show that

$$\overline{Z^n} = \int_0^1 d\mu\, M^{\frac{n}{\mu}}\, e^{N \frac{n}{\mu} g_\delta\left(\frac{\mu}{T}\right) + o(Nn)} \tag{5}$$

   where the order parameter $\mu$ has the same meaning as that discussed in the lectures. Let us denote with $\mu^*$ the saddle-point value of $\mu$, which optimizes the function at the exponent in (5).

   A. Using (5) and the replica trick, show that the quenched free energy is:

$$f_q(T) = -T \lim_{N\to\infty} \frac{1}{N} \overline{\log Z} = f_a\left(\frac{T}{\mu^*}\right). \tag{6}$$

B. Write the saddle point equation for $\mu$. Using this equation and the fact that $s_a(T) \equiv -\frac{df_a(T)}{dT}$, show that $\mu^*$ satisfies

$$\frac{1}{[\mu^*]^2}\, s_a\left(\frac{T}{\mu^*}\right) = 0. \tag{7}$$

Determine $\mu^*$ (hint: use the definition of $T_f$ also used in point 2). Is this consistent with what was discussed in the lecture?

## Problem 2: Interface: The Larkin model (7 points)

We focus on $d$ dimensional interfaces embedded in medium of $D = d + 1$ dimension. Their energy is

$$E_{\text{pot}} = \int d^d r \frac{1}{2}(\nabla h)^2 + V(h(r), r). \tag{8}$$

The first term represents the elasticity and the second the quenched disorder. Their competition originates a complex energy landscape with a single and non trivial global minimum.
Anatoly Larkin introduced a model where the disorder potential is linearized

$$V(h(r), r) \sim V(r) + F(r)h(r) \tag{9}$$

The Larking model has a trivial energy landscape, but plays a key role in understanding the problem of pinning. Here we ask you to solve it using the techniques that you learned.

(i) *Equation of motion. [2 pt]* Derive the equation of motion of the interface at zero temperature.

(ii) *Scaling Analysis. [2 pt]* Assume that $F(r)$ is a Gaussian with noise:

$$\overline{F(r)} = 0, \quad \overline{F(r)F(r')} = D\delta^d(r - r')$$

Predict the value of the roughness exponent $\zeta_L$ and the dynamics exponent $z_L$. Determine the upper critical dimension $d_{uc}$, above which the interface is flat.

(iii) *Center of mass. [1 pt]* For simplicity, consider a 1-dimensional interface of size $L$ with periodic boundary conditions. Assume that the interface is initially flat. Determine the mean displacement and the mean square displacement of the center of mass of the interface.

(iv) *Tagged monomer. [2 pt]* Compute now the mean square displacement of a tagged monomer $\overline{h^2(r,t)}$ in the co-moving frame of the center of mass. Comment about $\zeta_L$ and $z_L$.