Exercises 13-15: Branching, and Localization

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Exercise 13. Avalanches as branching processes

In this exercise we study a simple stochastic model describing avalanche dynamics. The model can be interpreted as a branching process with death.

The process is continuous in time. At time $t = 0$ a single site is unstable. Each unstable site can either recover (becoming stable) or destabilize another site.

We set the recovery rate to $a = 1$, while b denotes the rate at which an unstable site can destabilize a new one.

1. Explain why this process belongs to the class of *Bienaymé–Galton–Watson branching processes.*

2. Determine for which values of b the process is

   - supercritical,
   - critical,
   - subcritical.

   Give a physical interpretation in terms of avalanche propagation.

We now focus on $n(t)$, the number of unstable sites at time t. We denote by $P(n, t)$ its probability distribution and by $\langle n^k(t)\rangle$ its moments.

Consider the generating function

$$E(z, t) = P(0, t) + zP(1, t) + z^2 P(2, t) + \ldots = \sum_{n=0}^{\infty} z^n P(n, t). \tag{1}$$

3. Determine $E(z = 1, t)$ and $E(z, t = 0)$. What is the interpretation of $E(z = 0, t)$?

4. Express $\langle n(t)\rangle$ and $\langle n^2(t)\rangle$ in terms of derivatives of $E(z, t)$.

5. Write the time evolution equation of $E(z, t)$ using a *backward approach*. Specify the initial condition $E(z, 0)$.

6. Solve the differential equation for $E(z, t)$ in the **critical case** $b = 1$.

   A. Compute $\langle n(t)\rangle$ in the critical case.
   Hint: use the separation of variables

   $$\int_{E(z,0)}^{E(z,t)} \frac{dE}{(1 - E)^2} = \int_0^t dt.$$

   B. Compute $P(0, t)$, namely the probability that the entire system is stable at time t.

C. Compute the average number of unstable sites at time t *conditioned* on the system still being unstable, namely
$$\langle n(t) \rangle_{n(t)>0}.$$
Show that this quantity grows much faster than $\langle n(t) \rangle$ at large times.

7. The solution of the generating function equation for general $b \neq 1$ reads

$$E[z,t] = \left[\frac{b-1}{\frac{(z-1)e^{(b-1)t}}{bz-1} - 1} + b \right]^{-1}. \tag{2}$$

A. Verify that this expression satisfies the initial condition $E(z, 0)$.

B. Compute $\langle n(t) \rangle$ in the supercritical and subcritical regimes.
Discuss the behaviour of the avalanche in the two cases.

Exercise 14. The Landauer model for conductance

We consider electronic transport in one dimension ($d = 1$). Disorder is present only inside a finite region $(0, L)$. Outside this region the particle moves freely. The energy of the particle is

$$E = \frac{\hbar^2 k^2}{2m}. \tag{3}$$

The goal of this exercise is to derive the Landauer formula for the conductance and discuss its behavior in the localized regime.

Scattering states.

1. Construct the scattering states of energy E. Far from the disordered region the wavefunctions must reduce to plane waves.

   A state incoming from the left (with $k > 0$) behaves asymptotically as

$$\psi_{k,L}(x) = \begin{cases} e^{ikx} + r(E)e^{-ikx} & x \ll 0 \\ t(E)e^{ikx} & x \gg L (4) \end{cases}$$

Similarly, a state incoming from the right behaves as

$$\psi_{k,R}(x) = \begin{cases} e^{-ikx} + r(E)e^{ikx} & x \gg L \\ t(E)e^{-ikx} & x \ll 0 \end{cases} \tag{5}$$

Explain the physical meaning of the reflection coefficient $r(E)$ and transmission coefficient $t(E)$.

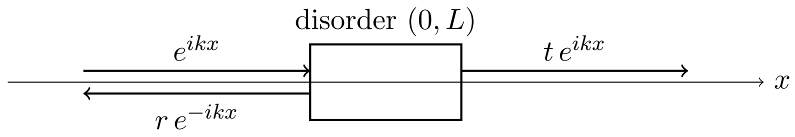

Figure 1: A scattering state is a solution of the Schrödinger equation on the whole line characterized by one incoming plane wave and outgoing waves after interaction with a potential in a bounded region.

2. The probability current associated to a wavefunction $\psi(x)$ in one dimension is

$$J = \frac{\hbar}{2mi}\left(\psi^* \frac{d\psi}{dx} - \psi \frac{d\psi^*}{dx}\right). \tag{6}$$

Compute the currents carried by the asymptotic plane waves (Recall that a plane wave e^{ikx} carries a probability current $J = \hbar k/m$) and show that

$$J_{k,L} = -J_{-k,R} = \frac{\hbar k}{m}|t(E)|^2. \tag{7}$$

Landauer current.

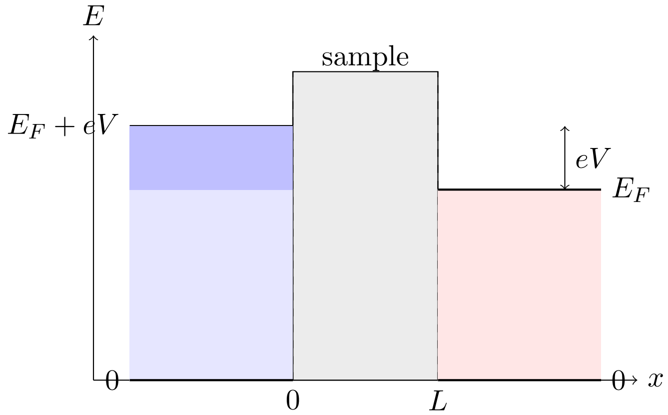

Figure 2: Energy diagram for the Landauer setup. The sample is confined to $(0, L)$. The energy spectrum is the same on both sides, but the reservoirs have different chemical potentials: on the right states are occupied up to E_F, while on the left they are occupied up to $E_F + eV$. Only the energy window of width eV contributes to the net current.

3. In the Landauer setup a voltage bias V is applied between two reservoirs. The net electric current is defined as

$$I = e\int_0^{E_F+eV} \frac{m\,dE}{\hbar^2 k}\, J_{k,L}(E) + e\int_0^{E_F} \frac{m\,dE}{\hbar^2 k}\, J_{-k,R}(E). \tag{8}$$

Where we used the Jacobian

$$\frac{\hbar^2 k}{m} dk = dE \tag{9}$$

Show that

$$I = \frac{e}{\hbar}\int_{E_F}^{E_F+eV} |t(E)|^2\, dE. \tag{10}$$

4. Assume now that the transmission varies slowly near the Fermi energy, so that

$$|t(E)|^2 \simeq |t(E_F)|^2 \tag{11}$$

for $E \in [E_F, E_F + eV]$. Show that the current becomes

$$I = \frac{e^2}{\hbar}|t(E_F)|^2 V. \tag{12}$$

Deduce the conductance

$$G = \frac{I}{V} = \frac{e^2}{\hbar}|t(E_F)|^2. \tag{13}$$

Localized regime.

5. In a one-dimensional disordered system the transmission typically decays exponentially with the system size:

$$|t(E_F)|^2 \sim \exp\left(-\frac{2L}{\xi_{\mathrm{loc}}}\right), \tag{14}$$

where ξ_{loc} is the localization length.

Using the Landauer formula, deduce the dependence of the conductance $G(L)$ on the system size L in the localized phase.

Exercise 15. Multiplicative random variables and self-averaging

Consider a set of positive i.i.d. random variables

$$x_1, x_2, \ldots, x_N$$

with finite mean and variance, and define their product

$$\Pi_N = \prod_{i=1}^{N} x_i, \qquad \text{or equivalently} \qquad \ln \Pi_N = \sum_{i=1}^{N} \ln x_i.$$

1. Using the Central Limit Theorem, show that for large N,

$$\ln \Pi_N = \gamma N + \gamma_2 \sqrt{N}\, \chi,$$

where χ is a Gaussian random variable with zero mean and unit variance.

Show that the coefficients are N-independent and given by

$$\gamma = \overline{\ln x}, \qquad \gamma_2 = \sqrt{\overline{(\ln x)^2} - (\overline{\ln x})^2}.$$

2. Deduce that Π_N follows a log-normal distribution and show that

$$P(\Pi_N)\, d\Pi_N = \frac{1}{\sqrt{2\pi\gamma_2^2 N}} \exp\left[-\frac{(\ln \Pi_N - \gamma N)^2}{2\gamma_2^2 N}\right] \frac{d\Pi_N}{\Pi_N}.$$

3. Introduce the variable

$$X = \ln \Pi_N,$$

which is Gaussian distributed with mean $\mu = \gamma N$ and variance $\sigma^2 = \gamma_2^2 N$.

Compute the moments

$$\overline{\Pi_N^n} = \int dX\, e^{nX} p(X),$$

and show that

$$\overline{\Pi_N^n} = \exp\left[\left(\gamma n + \frac{\gamma_2^2 n^2}{2}\right) N\right].$$

4. Compute explicitly the ratio

$$R_N = \frac{\overline{\Pi_N^2}}{\left(\overline{\Pi_N}\right)^2}.$$

Show that

$$R_N = \exp\left[\gamma_2^2 N\right].$$

5. Using this result, discuss whether Π_N is self-averaging.

Compare:

$$\overline{\Pi_N} = \exp\left[(\gamma + \gamma_2^2/2)N\right]$$

with the typical value

$$\Pi_N^{\text{typ}} = \exp(\gamma N).$$

Explain why Π_N is not self-averaging, while $\ln \Pi_N$ is.