Self-averagingness: A MATHEMATICAL NOTE

Two criteria for Y_N to be self-averaging

(1) $\lim_{N\to\infty} Y_N = \lim_{N\to\infty} \overline{Y_N} \equiv \overline{Y_\infty}$

(2) $\lim_{N\to\infty} \dfrac{\overline{Y_N^2} - (\overline{Y_N})^2}{(\overline{Y_N})^2} = 0$

The condition (1) implies (2). To argue that (2) implies (1), we can use Chebyshev's inequality:

$$\mathbb{P}\left(|Y_N - \overline{Y_N}| \geq \lambda \sqrt{\overline{Y_N^2} - (\overline{Y_N})^2} \right) \leq \lambda^{-2}.$$

that holds for any $\lambda > 0$

Let us choose $\lambda = \varepsilon \dfrac{\overline{Y_N}}{\sqrt{\overline{Y_N^2} - (\overline{Y_N})^2}}$

Then $$\mathbb{P}\left(|Y_N - \overline{Y_N}| \geq \varepsilon \overline{Y_N} \right) \leq \frac{1}{\varepsilon^2} \frac{\overline{Y_N^2} - (\overline{Y_N})^2}{(\overline{Y_N})^2}$$

For any arbitrary small ε, if

$$\lim_{N\to\infty} \frac{\overline{Y_N^2} - (\overline{Y_N})^2}{(\overline{Y_N})^2} = 0, \text{ then}$$

$$\lim_{N\to\infty} \mathbb{P}\left(|Y_N - \overline{Y_N}| \geq \varepsilon \overline{Y_N} \right) = 0,$$

meaning that deviation from the mean are suppressed with Probability 1, i.e. Y_N is no longer distributed in the limit $N \to \infty$, but it coincides with its mean.