SOLUTIONS TD9

Anderson on Bethe Lattice (2/2)

Problem 9: MOBILITY EDGE

$\boxed{1}$ IMAGINARY APPROXIMATION & DISTRIBUTIONAL EQUATIONS

Under all the approximation mentioned in the text,

$$\Gamma_a = V^2 \sum_{b \in \partial a} \frac{\Gamma_b + \eta}{\varepsilon_b^2 + (\Gamma_b + \eta)^2} \simeq V^2 \sum_{b \in \partial a} \frac{\Gamma_b + \eta}{\varepsilon_b^2} \qquad (*)$$

because we assume $\Gamma_b \simeq \eta \ll 1$.

Since, as remarked above, the Γ_b are all independent and identically distributed with density $P_\Gamma(\Gamma)$, the identity $(*)$ becomes a self consistent equation for $P_\Gamma(\Gamma)$.

In particular,

$$P_\Gamma(\Gamma) = \overline{\delta\left(\Gamma - V^2 \sum_{b \in \partial a} \frac{\Gamma_b + \eta}{\varepsilon_b^2}\right)}$$

which explicitly reads:

$$P_\Gamma(\Gamma) = \int \prod_{b=1}^{K} d\varepsilon_b \, p(\varepsilon_b) \int \prod_{b=1}^{K} d\Gamma_b \, P_\Gamma(\Gamma_b) \, \delta\left(\Gamma - v^2 \sum_{b=1}^{K} \frac{\Gamma_b + \eta}{\varepsilon_b^2}\right)$$

The Laplace transform is:

$$\Phi(s) = \int_0^\infty d\Gamma \, P_\Gamma(\Gamma) \, e^{-s\Gamma} = \int \prod_{b=1}^{K} d\Gamma_b \, P_\Gamma(\Gamma_b) \cdot \int \prod_{b=1}^{K} d\varepsilon_b \, p(\varepsilon_b) \, e^{-sv^2 \sum_{b=1}^{K} \frac{\Gamma_b + \eta}{\varepsilon_b^2}}$$

$\leftarrow$ independence

$$= \left[\int d\Gamma \, P_\Gamma(\Gamma) \int d\varepsilon \, p(\varepsilon) \, e^{-\frac{sv^2\Gamma}{\varepsilon^2} - \frac{sv^2\eta}{\varepsilon^2}} \right]^K$$

$$= \left[\int d\varepsilon \, p(\varepsilon) \, e^{-\frac{sv^2\Gamma}{\varepsilon^2}} \, \Phi\left(\frac{sv^2}{\varepsilon^2}\right) \right]^K$$

$\boxed{2}$ THE STABILITY ANALYSIS

- If $\Gamma \sim 1/\varepsilon^2$, then

$$P(\Gamma) \sim \int d\varepsilon \, p(\varepsilon) \, \delta\left(\Gamma - 1/\varepsilon^2\right) \sim \frac{p(\varepsilon)}{2} |\varepsilon|^3 \Big|_{\varepsilon = \Gamma^{-1/2}}$$

$$\sim 1/\Gamma^{3/2} \, P\left(\Gamma^{-1/2}\right) \overset{\Gamma \gg 1}{\sim} 1/\Gamma^{3/2}$$

- Assume $P_\Gamma(\Gamma) \sim \Gamma^{-\alpha}$ for $\alpha \in (1, 3/2]$.

First, it holds in full generality:

$$\lim_{s\to 0} \Phi(s) = \lim_{s\to 0} \int_0^\infty d\rho\, e^{-s\rho} P_\rho(\rho) = \int_0^\infty d\rho\, P_\rho(\rho) = 1$$

by normalization.

Consider $\left(\Phi(s) - 1\right) = \int_0^\infty d\rho\, P_\rho(\rho) \left(e^{-s\rho} - 1\right)$

Assume ρ has some dimension $[\rho]$.
Because the exponent has to be adimensional, $[s] = [\rho]^{-1}$.

Now, for $|s| \ll 1$ the integral is mostly contributed by $\rho \gg 1$, when $P_\rho(\rho) \sim \rho^{-\alpha}$. One has:

$$[\Phi(s) - 1] = [d\rho\, P_\rho(\rho)] = [\rho]^{1-\alpha} = [s]^{\alpha - 1}$$

Thus,

$$\Phi(s) = 1 - A|s|^{\beta} \qquad \beta = \alpha - 1$$
$$A > 0$$

(the sign is because $\Phi(s) \leq 1$)

- The equation for $\Phi(s)$ is:

$$\Phi(s) = \left[\int d\varepsilon \; p(\varepsilon) \, e^{-\frac{sV^2\eta}{\varepsilon^2}} \, \Phi\left(\frac{sV^2}{\varepsilon^2}\right) \right]^K$$

which implies: $(s>0)$

$$1 - A s^\beta = \left[\int d\varepsilon \; p(\varepsilon) \, e^{-\frac{sV^2\eta}{\varepsilon^2}} \left(1 - A \left|\frac{V}{\varepsilon}\right|^{2\beta} s^\beta \right) \right]^K$$

$$= \left[1 + O(s) - A s^\beta \int d\varepsilon \; \underbrace{p(\varepsilon)}_{\text{small}} \, e^{-sV^2\eta/\varepsilon^2} \left|\frac{V}{\varepsilon}\right|^{2\beta} \right]^K$$

$$\simeq 1 - A K s^\beta \int d\varepsilon \; p(\varepsilon) \left|\frac{V}{\varepsilon}\right|^{2\beta} + o(s^\beta)$$

The two sides match provided that $\exists \beta$:

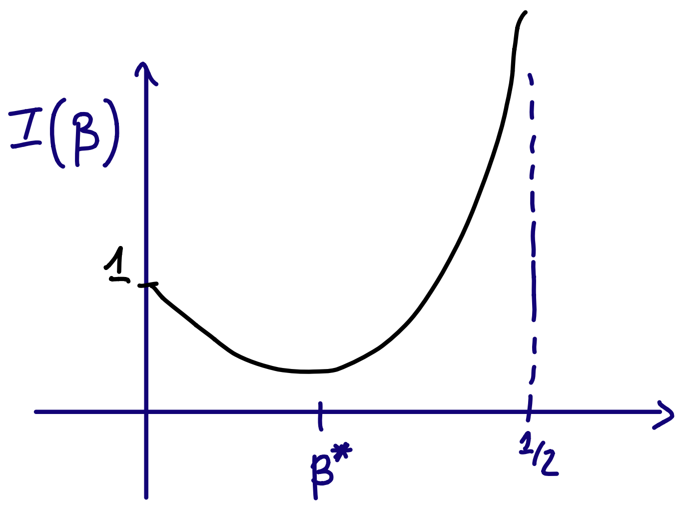
$$\int d\varepsilon \; p(\varepsilon) \left(\frac{V}{|\varepsilon|}\right)^{2\beta} = 1/K .$$

3 CRITICAL DISORDER

For the uniform distribution:

$$I(\beta) = \frac{1}{W}\int_{-W/2}^{W/2} d\varepsilon \left|\frac{V}{\varepsilon}\right|^{2\beta} = \left(\frac{V}{W}\right)^{2\beta}\int_{-1/2}^{1/2} d\varepsilon \; 1/|\varepsilon|^{2\beta}$$

$$= \left(\frac{2V}{W}\right)^{2\beta}\frac{1}{1-2\beta} = \frac{e^{2\beta\log(2V/W)}}{1-2\beta}$$



$$I(\beta) \xrightarrow{\beta=0} 1$$

$$I(\beta) \xrightarrow{\beta=1/2} \infty$$

$$I'(\beta) = 2\log\left(\frac{2V}{W}\right)I(\beta) + \frac{2\cdot I(\beta)}{1-2\beta} = 2I(\beta)\left[\log\left(\frac{2V}{W}\right) + \frac{1}{1-2\beta}\right]$$

This vanishes when $\beta = \beta^* = \left(\frac{1}{\log\left(\frac{2V}{W}\right)} + 1\right)\frac{1}{2}$

And:

$$I(\beta^*) = \left[e^{\log\left(\frac{2V}{W}\right)+1}\right]\left(-\log\left(\frac{2V}{W}\right)\right) = \frac{2Ve}{W}\log\left(\frac{W}{2V}\right)$$

when W decreases, $I(\beta^*)$ increases: eventually, for W small enough it will reach $1/K$ (where $K \geq 2$), & localization becomes unstable

In particular, this happens when:

$$\frac{2Ve}{W_c} \log\left(\frac{W_c}{2V}\right) = \frac{1}{K} \implies W_c = K\, 2Ve \log\left(\frac{W_c}{2V}\right)$$

Iterating and assuming $K \gg 1$

$$\left(\frac{W_c}{V}\right) = K\, 2\, e \log(K).$$

The critical disorder increases with the connectivity K of the graph: when K is larger, there are more "directions" along which the particle can move, and one needs a stronger disorder to localize it. Increasing K is like increasing dimensionality: localization becomes more difficult, i.e. it requires a stronger disorder.