Problem 1: the energy landscape of the REM

1 ANNEALED ENTROPY

We write: $\mathcal{N}_N(E)dE = \sum_{\alpha=1}^{2^N} \chi_\alpha(E)\, dE$ $\qquad \chi_\alpha(E) = \begin{cases} 1 & E_\alpha \in [E, E+dE] \\ 0 & \text{otherwise} \end{cases}$

Taking the average:

$$\overline{\mathcal{N}_N(E)}\, dE = \sum_{\alpha=1}^{2^N} \overline{\chi_\alpha(E)}\, dE = \sum_{\alpha=1}^{2^N} P\Big(E_\alpha \in [E, E+dE]\Big) =$$

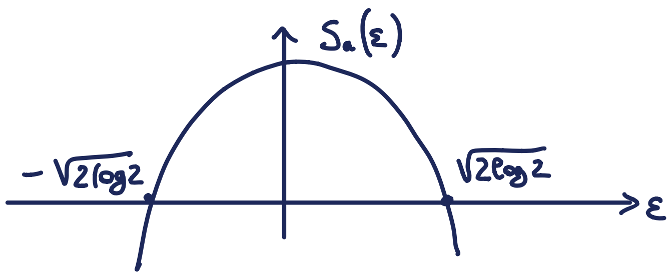
$$= \sum_{\alpha=1}^{2^N} p(E)\, dE = 2^N \frac{1}{\sqrt{2\pi N}} e^{-E^2/2N}\, dE$$

$$= e^{N\left(\log 2 - \frac{1}{2}(E/N)^2\right) + o(N)}\, dE$$

Introducing the energy density $\varepsilon = E/N$ we have

$$S_a(\varepsilon) = \lim_{N\to\infty} \frac{1}{N} \log \overline{\mathcal{N}_N}(\varepsilon) = \log 2 - \varepsilon^2/2 .$$

This function is plotted below:



2 SELF-AVERAGING

Let us compute the second moment, as above:

$$\overline{\mathcal{N}_N^2(E)} = \sum_{\alpha=1}^{2^N}\sum_{\beta=1}^{2^N} \overline{\chi_\alpha(E)\chi_\beta(E)} \underset{\text{independence}}{\overset{\uparrow}{=}} \sum_{\substack{\alpha,\beta=1\\(\alpha\neq\beta)}}^{2^N} \overline{\chi_\alpha(E)}\;\overline{\chi_\beta(E)} +$$

$$+ \sum_{\alpha=1}^{2^N} \overline{\chi_\alpha^2(E)} = \sum_{\alpha=1}^{2^N} \overline{\chi_\alpha(E)}\left[\sum_{\beta:\beta\neq\alpha}\overline{\chi_\beta} + 1\right]$$

$$= \overline{\mathcal{N}_N(E)}\left[(2^N-1)\frac{e^{-E^2/2N}}{\sqrt{2\pi N}} + 1\right] =$$

$$= \overline{\mathcal{N}_N(E)}\left[\overline{\mathcal{N}_N(E)} + \left(1 - \frac{e^{-E^2/2N}}{\sqrt{2\pi N}}\right)\right]$$

Therefore:

$$\frac{\overline{\mathcal{N}_N^2(E)}}{\left(\overline{\mathcal{N}_N(E)}\right)^2} = 1 + \frac{1}{\overline{\mathcal{N}_N(E)}}\left(1 - \frac{e^{-E^2/2N}}{\sqrt{2\pi N}}\right)$$

And $\overline{\mathcal{N}_N(E)} = e^{N S_a(E/N) + o(N)}$

when $S_a > 0$, $\overline{\mathcal{N}_N(E)}$ grows exponentially and

$$\lim_{N\to\infty} \frac{\overline{\mathcal{N}_N^2(E)}}{\left(\overline{\mathcal{N}_N(E)}\right)^2} = 1 \implies$$

$\mathcal{N}_N(E)$ is self-averaging, thus $S_a(\varepsilon) = S(\varepsilon)$ for $|\varepsilon| \leq \sqrt{2\log 2}$

For $|\varepsilon| > \sqrt{2\log 2}$ the average decays to zero faster than the standard deviation: the fluctuations are not negligible and the large-N behavior is not controlled by the average.
The quantity is not self-averaging: sample-to-sample fluctuations will matter when N is large.

3 AVERAGE VS TYPICAL

Let us try to bound the probability to have configurations with $|\varepsilon| > \sqrt{2\log 2}$. It holds:

$$\mathbb{P}\left(\begin{array}{c}\text{exists at least one configuration} \\ E_\alpha \text{ such that } E_\alpha \in [E, E+dE] \\ \text{with } E < -N\sqrt{2\log 2}\end{array}\right) =$$

$$= \sum_{n=1}^{2^N} \mathbb{P}\left(\begin{array}{c}\text{exist } n \text{ configurations} \\ \text{in } [E, E+dE]\end{array}\right) \leq \sum_{n=1}^{2^N} n\,\mathbb{P}\left(\begin{array}{l}\text{exist } n \\ \text{configs in } [E, E+dE]\end{array}\right)$$

$$= \overline{\mathcal{N}_N(E)} \Rightarrow \text{exponentially small.}$$

Notice: this bound is a special case of Markov's inequality, $\mathbb{P}(X \geq a) \leq \mathbb{E}[X]/a$.

applied to the random variable $X = \mathcal{N}_N(E)$ and $a = 1$.

Since the probability to find a configuration is exponentially small in this region, the <u>typical</u> number of configurations is zero: $N^{typ}(\varepsilon)$ vanishes, and so $S(\varepsilon) = -\infty$.

Thus, putting everything together we have:

$$S(\varepsilon) = \begin{cases} S_a(\varepsilon) = \log 2 - \varepsilon^2/2 & |\varepsilon| \leq \sqrt{2\log 2} \\ -\infty & |\varepsilon| > \sqrt{2\log 2} \end{cases}$$

The point where $S(\varepsilon)=0$ gives the minimal (maximal) energy density at which one finds configurations with a probability that is $\mathcal{O}(1)$ as $N \to \infty$: it defines the TYPICAL VALUE of the ground state energy density. This is consistent with what found in lecture 1 by Alberto on extreme values:

$$E_{min}^{typ} = a_M + b_M z$$

$$a_M = -N\sqrt{2\log 2}$$