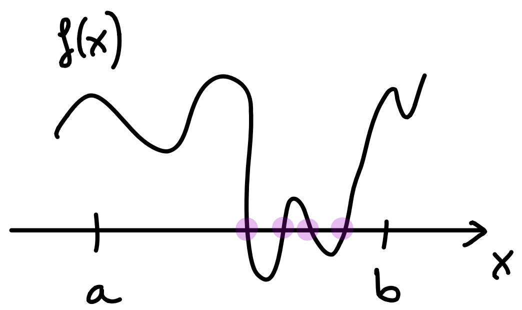
SOLUTIONS TD5

Landscapes & Kac-Rice (1/2)

Problem 5 : The KR Method & The Complexity

1 The Kac-Rice formula



One can write

$$\mathcal{N} = \int_a^b dx\, \delta\left(x - f^{-1}(0)\right)$$

Using the formula of change of variables of $\delta(\cdot)$,

$$\delta\left(y - F(x)\right) = \frac{1}{|F'(x)|}\, \delta\left(x - F^{-1}(y)\right)$$

Jacobian

we have $\mathcal{N} = \int_a^b dx\, \delta(f(x))\, |f'(x)|$ and the formula is obtained taking the average.

Now, we want to count the number of points $\vec{\sigma} \in S_N$

such that $f(x) \equiv \nabla_\perp E(\vec{\sigma}) = 0$. Since this function is a vector, its derivative has to be replaced by a matrix, the Hessian: the modulus is replaced by the absolute value of the determinant.

Therefore, the Kac-Rice formula counting the total number of stationary points is:

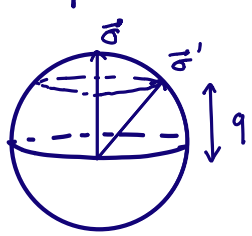
$$\overline{\mathcal{N}} = \int_{S_N} d\vec{\sigma}\ \overline{\delta\left(\nabla_\perp E(\vec{\sigma})\right) \left|\det \nabla_\perp^2 E(\vec{\sigma})\right|}$$

To count only those having a given value of energy density, one adds the constraint: $\delta\left(E(\vec{\sigma}) - N\varepsilon\right)$.

2 STATISTICAL ROTATIONAL INVARIANCE

The correlations read: $\overline{E(\vec{\sigma})\,E(\vec{\sigma}')} = N\left(\frac{\vec{\sigma}\cdot\vec{\sigma}'}{N}\right)^p$

Given two configurations on the sphere, the correlation between the corresponding energies depends only on the overlap between the configurations



The overlap $q = \vec{\sigma}\cdot\vec{\sigma}'/N$ is rotational invariant, and so are the correlations.

The Kac-Rice formula reads:

$$\overline{\mathcal{N}(\varepsilon)} = \int_{S_N} d\vec{\sigma} \; p_\varepsilon(\vec{\sigma})$$

The fact that the correlations, and thus the statistics of $E(\vec{\sigma})$, is rotational invariant implies that $p_\varepsilon(\vec{\sigma})$ does not depend on the particular configuration $\vec{\sigma}$, but it is the same for all $\vec{\sigma}$. We fix one value for the vector, $\vec{1} = (1, \ldots, 1)$, and get:

$$\overline{\mathcal{N}(\varepsilon)} = \int_{S_N} d\vec{\sigma} \; p_\varepsilon(\vec{\sigma}) = p_\varepsilon(\vec{1}) \int_{S_N} d\vec{\sigma} = p_\varepsilon(\vec{\sigma}) \underbrace{(2\pi e)^{N/2}}_{\text{volume of sphere}}$$

3 GAUSSIANITY & CORRELATIONS

- The quantity $E(\vec{1}) = \sum_{i_1 < \ldots < i_p} J_{i_1 \ldots i_p}$ is a sum of independent Gaussian variables, and thus it is a gaussian variable with zero mean and:

  $$\overline{(E(\vec{1}))^2} = N$$

  Its distribution is $\frac{dE}{\sqrt{2\pi N}} e^{-E^2/2N}$; in terms of $\varepsilon = E/N$, this is $\sqrt{\frac{N}{2\pi}} \, d\varepsilon \, e^{-\frac{N\varepsilon^2}{2}}$

- The vectors $\nabla_\perp E(\vec{1})$ are obtained subtracting to $\nabla E(\vec{1})$ the component parallel to $\vec{1}$: one remains with $(N-1)$ components that are gaussian, with covariances:

$$\overline{(\nabla_\perp E)_\alpha (\nabla_\perp E)_\beta} = p\,\delta_{\alpha\beta}$$ and with zero average.

The probability density is:

$$\mathcal{P}(\nabla_\perp E) = \frac{1}{(2\pi p)^{\frac{N-1}{2}}} \prod_{\alpha=1}^{N-1} e^{-\frac{(\nabla_\perp E)_\alpha^2}{2p}}$$

which computed at $\nabla_\perp E = 0$ gives $\mathcal{P}(\vec{0}) = \left(\frac{1}{2\pi p}\right)^{\frac{N-1}{2}}$

- Since $\nabla_\perp E(\vec{1})$ is independent of $E(\vec{1}), \nabla_\perp^2 E(\vec{1})$, we have

$$\overline{\left|\det \nabla_\perp^2 E(\vec{1})\right| \cdot \delta\left(\nabla_\perp E(\vec{1})\right) \cdot \delta\left(E(\vec{1}) - N\varepsilon\right)} =$$

$$= \overline{\left|\det \nabla_\perp^2 E(\vec{1})\right| \delta\left(E(\vec{1}) - N\varepsilon\right)} \cdot \overline{\delta\left(\nabla_\perp E(\vec{1})\right)}$$

$$= \overline{\left|\det \nabla_\perp^2 E(\vec{1})\right| \delta\left(E(\vec{1}) - N\varepsilon\right)} \cdot \mathbb{P}\left(\nabla_\perp E(\vec{1}) = 0\right)$$

$$= \overline{\left|\det \nabla_\perp^2 E(\vec{1})\right| \delta\left(E(\vec{1}) - N\varepsilon\right)} \cdot \left(\frac{1}{2\pi p}\right)^{\frac{N-1}{2}}$$

- Finally, the average of the determinant can be written as a conditional average, using the general relation:

$$\overline{X\cdot\delta(Y-y)} = \int dx\, dy'\, \underbrace{P(x,y')}_{P(x|y')\,p(y')} \cdot x \cdot \delta(y'-y) = \left(\int dx\, P(x|y)\cdot x\right)\cdot p(y)$$

$$= \overline{X|(Y=y)}\;\; \mathbb{P}(Y=y)$$

conditional average

Therefore:

$$\overline{\left|\det \nabla_\perp^2 E(\vec{1})\right|\,\delta\big(E(\vec{1})-N\varepsilon\big)} = \overline{\left|\det \nabla_\perp^2 E(\vec{1})\right| \,\Big|\, E(\vec{1})=N\varepsilon} \cdot \mathbb{P}(E=N\varepsilon)$$

conditional average

Using the fact that the conditional law of the Hessian matrix $\nabla_\perp^2 E(\vec{1})$ is the same as matrices $(M-p\varepsilon\mathbb{1})$, where M is symmetric with Gaussian entries, we have that this term reduces to:

$$\overline{\left|\det \nabla_\perp^2 E(\vec{1})\right|\,\delta\big(E(\vec{1})-N\varepsilon\big)} = \overline{\left|\det(M-p\varepsilon\mathbb{1})\right|}\cdot\sqrt{\frac{N}{2\pi}}\, e^{-\frac{N\varepsilon^2}{2}}$$

Combining everything:

$$\overline{N(\varepsilon)} = (2\pi e)^{N/2} \left(\frac{1}{2\pi p}\right)^{\frac{N-1}{2}} \sqrt{\frac{N}{2\pi}}\, e^{-\frac{N\varepsilon^2}{2}} \, \overline{\left|\det\left(M - p\varepsilon \mathbb{1}\right)\right|}^{M}$$

$$= e^{\frac{N}{2}\log\left(\frac{e}{p}\right) - \frac{N}{2}\varepsilon^2 + \log \overline{\left|\det\left(M - p\varepsilon\mathbb{1}\right)\right|} + o(N)},$$

where the average of the determinant is over the ensemble of matices M that are symmetic, and have independent random entries $M_{\alpha\beta}$ with $\alpha \leq \beta$, with aussian statistics.