PROBLEMS 7
Trap model & aging

Problem 7.1: A SIMPLE MODEL FOR AGING

$\boxed{1}$ CONDENSATION & ERGODICITY BREAKING

- The average trapping time is:

$$\overline{\tau} = \int_{\tau_0}^{\infty} d\tau \; \tau \cdot P_\mu(\tau) = \mu \, \tau_0^{\mu} \int_{\tau_0}^{\infty} d\tau \; \tau^{-\mu} = \frac{\mu \tau_0}{\mu - 1} \qquad \text{if } \mu > 1$$

If $\mu \leq 1$, the first moment of the power-law distribution does not exist: the average trapping time diverges. ERGODICITY BREAKING!

- Consider the trap-like dynamics from t_w to $t_w + t$. Assume that in this time interval the system has visited $n(t)$ traps.

We assume that the time spent in each trap α is exactly τ_α (the average time).

Then: $t = \sum_{\alpha=1}^{n(t)} \tau_\alpha$, which is a sum of $n(t)$ independent random variables, distributed as a power-law with exponent μ.

We look at the maximum between these $n(t)$ values.

Recall from Lecture 1 that the typical value of the maximum, τ_{max}^{typ}, satisfies:

$$P\left(\tau \geq \tau_{max}^{typ}\right) = \int_{\tau_{max}^{typ}}^{\infty} d\tau \, P_m(\tau) = \frac{1}{n(t)} \Rightarrow \frac{\tau_0^\mu}{\left(\tau_{max}^{typ}\right)^\mu} = \frac{1}{n(t)}$$

Which gives $\tau_{max}^{typ}(t) = \tau_0 \left[n(t)\right]^{1/\mu}$

Now there are two cases:

$\boxed{\mu > 1}$ There exists a finite $\overline{\tau}$. By the law of large numbers, one has that for $t \gg 1$:

$$\frac{t}{n(t)} = \frac{1}{n(t)} \sum_{\alpha=1}^{n(t)} \tau_\alpha \xrightarrow{t \gg 1} \overline{\tau} \Longrightarrow n(t) = t/\overline{\tau} = \frac{\mu-1}{\mu}\left(t/\tau_0\right)$$

which suggests that the the time spent in each trap is of the order of τ_0, including the maximal time. Indeed, plugging this in the formula:

$$\tau_{max}^{typ}(t) = \tau_0 \left[\frac{(\mu-1)}{\mu} t/\tau_0\right]^{1/\mu}$$

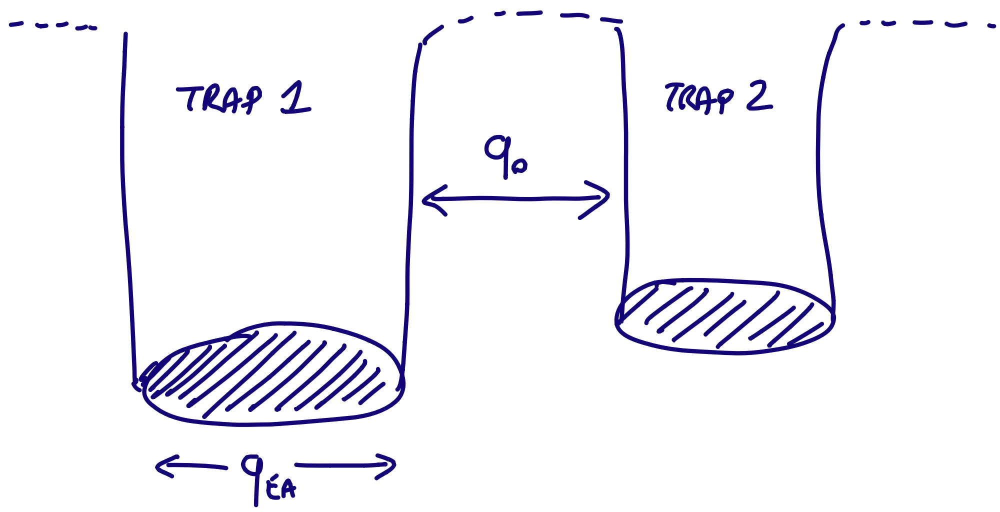
$\boxed{\mu \leq 1}$ In this case, the law of large numbers is violated and the sum scales as the maximum:

$$t = \sum_{\alpha=1}^{N(t)} \tau_\alpha = \tilde{C}_\mu [n(t)]^{1/\mu} \Longrightarrow n(t) = t^\mu C_\mu$$

$$\Longrightarrow \tau_{max}^{typ}(t) = \alpha_\mu \cdot t$$

The maximal time spent in a trap is of the order of the total time! This is a condensation phenomenon: the system spends <u>a finite fraction</u> of the whole time in a single trap, the one with the largest trapping time among all those it has encountered in the interval.

2 AGING & WEAK ERGODICITY BREAKING



The correlation $C(t_w+t, t_w)$ measures the overlap between the configurations at time t_w and t_w+t. There are two possibilities: either at the two times the system is in the same trap and $C = q_{EA}$, or the system is in two $\neq$ traps and $C = q_0$.

To be in the same trap at time t_w and t_w+t, the system has to be stayed there in the whole time interval, which happens with probability Π.

Notice that we are neglecting the situations in which the system jumps out of a trap and subsequently falls back in the same: this is because these events are suppressed in $M \gg 1$.

- The correlation depends on time only through the rescaled quantity $t/_{t+t_w}$, i.e. t over the age of the system $t+t_w$.

- $\lim_{t\to\infty} C(t_w+t, t_w) = q_0 = 0$ $\quad$ $\lim_{t_w\to\infty} C(t_w+t, t_w) = q_{EA}$

tw finite

If I fix the observation time tw and I give to the system infinite time, it will eventually escape from the trap and decorrelate.

However, if I let the system evolve and age ($t_w \to \infty$), it will encounter deeper and deeper traps and in any finite time interval t it will not be able to escape.

3 (EXTRA): POWER LAWS

- Asymptotics $t \ll t_w$

In this case, $t/t+t_w := \varepsilon \ll 1$

Then:

$$\int_\varepsilon^1 du\,(1-u)^{\mu-1} u^{-\mu} = \underbrace{\int_0^1 du\,(1-u)^{\mu-1} u^{-\mu}}_{\frac{\pi}{\sin(\pi\mu)}} - \underbrace{\int_0^\varepsilon du\,(1-u)^{\mu-1} u^{-\mu}}_{\text{by parts}}$$

by parts

$$(1-u)^{\mu-1}\frac{u^{-\mu+1}}{1-\mu}\Big|_0^\varepsilon + \int_0^\varepsilon du\, u^{-\mu+1}(1-u)^{\mu-2}$$

$$\simeq \frac{\pi}{\sin(\pi\mu)} - \frac{1}{1-\mu}\varepsilon^{1-\mu} + o(\varepsilon^{1-\mu})$$

Then $\Pi(t_w+t, t_w) \overset{t \ll t_w}{\approx} 1 - \frac{\sin(\pi\mu)}{\pi(1-\mu)}\left(\frac{t}{t+t_w}\right)^{1-\mu}$

- Asymptotics $t \gg t_w$

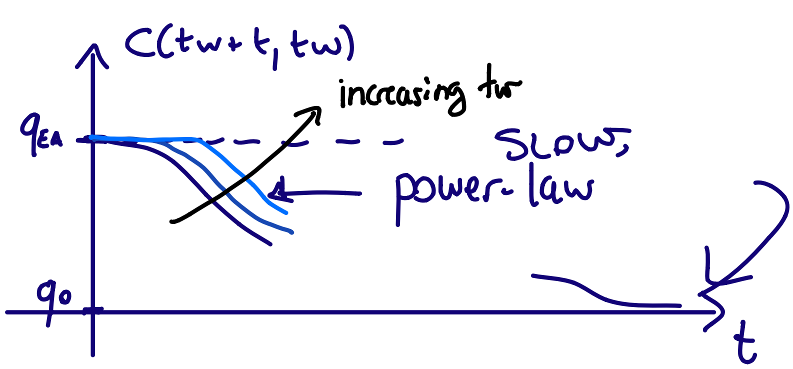
In this case, $t/_{t+t_w} = 1-\varepsilon$ with $\varepsilon \ll 1$

$$\int_{1-\varepsilon}^{1} du\, \underbrace{(1-u)}_{\text{small}}{}^{\mu-1}\, u^{-\mu} \simeq -\frac{(1-u)^{\mu}}{\mu}\, u^{-\mu}\Big|_{1-\varepsilon}^{1} + \int_{1-\varepsilon}^{1} (1-u)^{\mu}\, u^{-\mu+1}$$

$$\simeq \frac{\varepsilon^{\mu}}{\mu}(1-\varepsilon)^{-\mu} + o(\varepsilon^{\mu})$$

and $\Pi(t_w+t, t_w) \approx \frac{\sin(\pi\mu)}{\pi\mu}\left(1-\frac{t}{t+t_w}\right)^{\mu} = \frac{\sin(\pi\mu)}{\pi\mu}\left(\frac{t_w}{t+t_w}\right)^{\mu}$

Therefore at fixed t_w



Problem 7.2: From Landscapes to Traps

1 REM: Distribution of depths of traps

- In the REM, the ground state has energy $E_{GS}^{typ} = -N\sqrt{2\log 2}$.
  all configurations having the same energy up to $\mathcal{O}(1)$ corrections are extreme values: their distribution is a Gumbel Distribution, as derived in lecture 1.

  More precisely, the fluctuations of the ground state around its typical value scale as:

  $\left(E_{GS}^{typ} - E\right) \sim b_M = \dfrac{1}{\sqrt{2\log 2}}$, which is also the typical separation between the levels above the ground state.
  Moreover, the variable:

  $z = \sqrt{2\log 2}\left(E_{GS}^{typ} - E\right)$ is distributed as $p(z) = e^{-\left(z + e^{-z}\right)}$.

  This implies that E is distributed as

$$P_N^{extrm}(E) = \int dz \; p(z)\; \delta\left(E - E_{GS}^{typ} + \frac{z}{\sqrt{2\log 2}}\right)$$

$$\propto e^{\sqrt{2\log 2}\,(E + N\sqrt{2\log 2}) - \exp\left[\sqrt{2\log 2}\,(E + N\sqrt{2\cdot\log 2})\right]}$$

For $(E+N\sqrt{2\log 2}) \gtrsim 1$, this is well approximated by an exponential, $e^{-z-e^{-z}} \simeq e^{-z}$, and one recovers the iven distribution.

2 REM: TRAPPING TIMES

Escaping from a deep minimum with energy density $\varepsilon<0$ to energy $\varepsilon=0$ requires

$$\tau \sim e^{-\beta N \varepsilon} \sim e^{\beta |E|}.$$

The distribution of trapping times is obtained through a change of variable:

$$P_\mu(\tau) = \int dE \; P_N^{\text{extrm}}(E)\, \delta\left(\tau - e^{-\beta E}\right) = \frac{1}{\beta} e^{\beta E} P_N^{\text{extr}}(E) \Big|_{E=-\frac{\log \tau}{\beta}} =$$

$$= \tilde{A}_N \frac{1}{\tau^{1+\mu}}$$

normalization

$$\boxed{\mu = \frac{\sqrt{2\log 2}}{\beta}.}$$

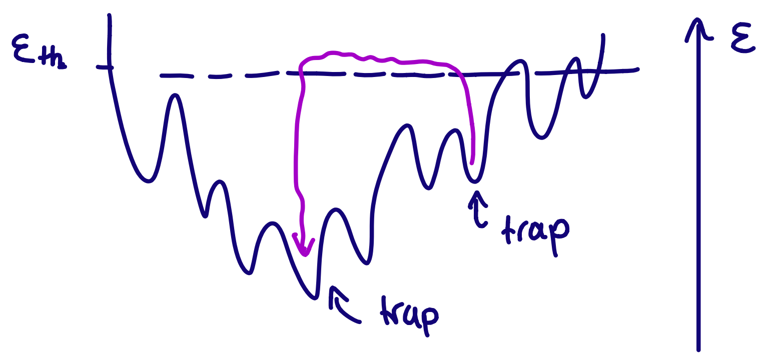
This has the same form as in the trap model. The non-ergodic phase is for $\mu<1$, meaning:

$$T \leq \frac{1}{\sqrt{2\log 2}} = T_g \quad (\text{freezing})$$

Which is precisely the freezing transition temperature of the REM.
Thus, a trap model with $\mu < 1$ describes the dynamics in the glass phase of the REM

3 EXTRA: P-SPIN AND THE "TRAP" PICTURE

- A trap can be identified with one of the local minima in the energy landscape of the p-spin:



In the trap model, one always assumes that the trapping time depends only on the energy of the trap.
However, in the Arrhenius law the energy depends on the barrier to be crossed: the barrier can be in general a function of the energy of the minimum itself.
When using a trap-like description, we are assuming that this is not the case: the level to be reached to escape from a minimum is the same for each minimum. In the REM, it is $\varepsilon = 0$. In the p-spin, it can be identified with the threshold energy.