Rice method for the extremes of Gaussian fields

Saclay, June 15 2011

Jean-Marc Azaïs

Institut de Mathématiques, Université de Toulouse

Examples

The record method $d = 2$ or 3

The maxima method

Second order

Processes defined on fractal sets

Signal + noise model

Spatial Statistics often uses " signal + noise model", for example :

- precision agriculture
- neuro-sciences
- sea-waves modelling

Signal + noise model

Spatial Statistics often uses “ signal + noise model”, for example :

- precision agriculture
- neuro-sciences
- sea-waves modelling

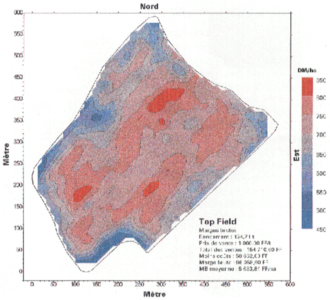
Signal + noise model

Spatial Statistics often uses “ signal + noise model”, for example :

- precision agriculture
- neuro-sciences
- sea-waves modelling

Precision agriculture

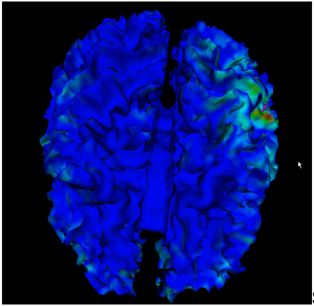
Representation of the yield per unit by GPS harvester .



Is there only noise or some region with higher fertility ? ?

Neuroscience

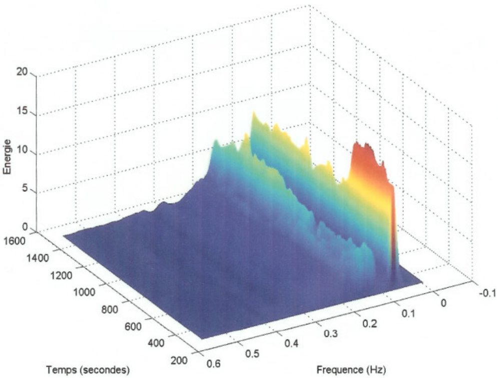
The activity of the brain is recorded under some particular action and the same question is asked



source : Maureen CLERC

Sea-waves spectrum

Locally in time and frequency the spectrum of waves is registered.
We want to localize transition periods.



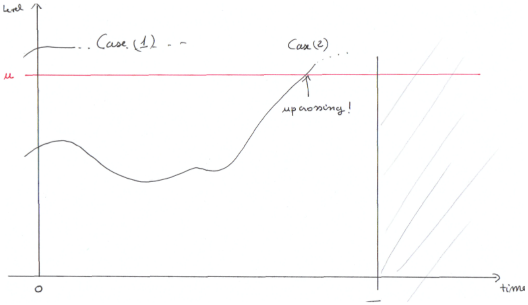
In all these situations a good statistics consists in observing the **maximum** of the (absolute value) of the random field for deciding if it is **typically** (Noise) or **too large** (signal).

The Rice method

M is the maximum of a smooth random process $X(t), t \in \mathbb{R}^d$ $(d = 1)$ or field $(d > 1)$.
We want to evaluate

$$\mathbb{P}\{M > u\}$$

In dimension 1 : count the number of up-crossing



The Rice formula

If $X(t)$ is a regular (differentiable) process $\mathbb{R} \to \mathbb{R}$ or a random field $\mathbb{R}^d \to \mathbb{R}^d$ and if we consider its number of zeros :

$$N_u = \#\{t \in [0,T] : X(t) = u\}(d = 1)$$

We obtain a random variable. In general nothing is known except for the moments of this variable.

For the expectation, for random processes $(d = 1)$ we get the simplest version of the Rice formula :

$$\mathrm{E}(N_u) = \int_0^T \mathrm{E}(|X'(t)| \big| X(t) = u) p_{X(t)}(u),$$

where p is the density
The proof of this formula is based on some generalization of the change of variable formula. It explains the term $|X'(t)|$.
For $X : \mathbb{R}^d \to \mathbb{R}^d$, $d > 1$ we must put $|\det(X'(t)|$.

In larger dimension search for an other geometrical characteristic

- number of connected component of the excursion set :open problem
- Euler characteristic an alternative to the preceding : Conceptually complicated, computationally easy no bounds
- Number of maxima above the considered level : difficult to compute the determinant but gives bounds
- Particular points on the level set : the record method

In larger dimension search for an other geometrical characteristic

- number of connected component of the excursion set :open problem
- Euler characteristic an alternative to the preceding : Conceptually complicated, computationally easy no bounds
- Number of maxima above the considered level : difficult to compute the determinant but gives bounds
- Particular points on the level set : the record method

In larger dimension search for an other geometrical characteristic

- number of connected component of the excursion set :open problem
- Euler characteristic an alternative to the preceding : Conceptually complicated, computationally easy no bounds
- Number of maxima above the considered level : difficult to compute the determinant but gives bounds
- Particular points on the level set : the record method

In larger dimension search for an other geometrical characteristic

- number of connected component of the excursion set :open problem
- Euler characteristic an alternative to the preceding : Conceptually complicated, computationally easy no bounds
- Number of maxima above the considered level : difficult to compute the determinant but gives bounds
- Particular points on the level set : the record method

The maxima method

Let us forget the boundary

$$\{M > u\} = \text{ There is a maximum above } u =: \{M_u > 0\}$$

$$\mathbb{P}\{M > u\} \leq \mathrm{E}(M_u)$$
$$= \int_u^{+\infty} dx \int_S \mathrm{E}\Big[\big|\det(X''(t))\ \mathbf{I}_{X''(t) \prec 0}\big| \,\Big|\, X(t) = x, X'(t) = 0\Big] P_{X(t),X'(t)}(x,0)dt$$

Very difficult to compute : the expectation of absolute value of the determinant

Examples

The record method $d = 2$ or 3

The maxima method
Second order

Processes defined on fractal sets

Under some conditions, Roughly speaking the event

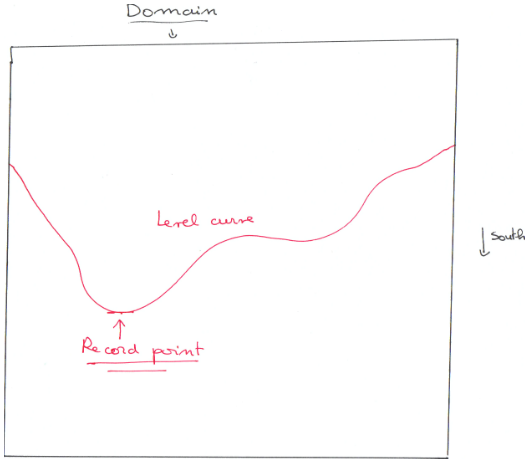
$$\{M > u\}$$

is almost equal to the events

"The level curve at level u is non-empty"

"The point at the southern extremity of the level curve exists"

"There exists a point on the level curve :
$X(t) = u; X_1'(t) = \frac{\partial X}{\partial t_1} = 0; X_2'(t) = \frac{\partial X}{\partial t_2} > 0$
$X_{11}''(t) = \frac{\partial^2 X}{\partial t_1^2} < 0$"


Domain
Level curve
Record point
South

Forget the boundary

and define

$$Z(t) := \begin{pmatrix} X(t) \\ X'_1(t) \end{pmatrix}$$

The probability above is bounded by the expectation of the number of roots of $Z(t) - (u, 0) \Rightarrow$ Rice formula

$$\mathbb{P}\{M > u\} \leq \text{Boundary terms}$$

$$+ \int_S \mathrm{E}(|\det(Z'(t)\ \mathbf{I}_{X''_1(t)<0}\ \mathbf{I}_{X'_2(t)>0}|\big|X(t) = u, X'_1(t) = 0) p_{X(t),X'_1(t)}(u, 0) dt,$$

The difficulty lies in the computation of the expectation of the determinant

The trick is that under the condition $\{X(t) = u, X_1'(t)\} = 0$, the quantity

$$|\det(Z'(t)| = \begin{vmatrix} X_1' & X_2' \\ X_{11}'' & X_{12}'' \end{vmatrix}$$

is simply equal to $|X_2' X_{11}''|$. Taking into account conditions, we get the following expression for the second integral

$$\int_S \mathrm{E}(|X_{11}''(t)^- X_2'(t)^+ \big| X(t) = u, X_1'(t) = 0) p_{X(t),X_1'(t)}(u,0) dt.$$

Moreover under stationarity or some more general hypotheses, these two random variables are independent.

Theorem

Suppose that the set S is the square $[0,T]^2$ and that the process is stationary isotrope with $\mathrm{E}(X(t) = 0$,$var(X(t) = 1$, $var(X'(t) = Id$ and satisfies some regularity conditions.
Then

$$\mathbb{P}\{M > u\} \leq \overline{\Phi}(u) + \sqrt{2/\pi}T\phi(u) + T^2/(2\pi)[c\phi(u/c) + u\Phi(u/c)]\phi(u)$$

Extension to dimension 3

Using Fourier method (Li and Wei 2009) we are able to compute the expectation of the absolute value of a determinant in dimension 2.
It is a simple quadratic form.
We are able to extend the result to dimension 3 (Pham 2010).

$$\mathbb{P}\{M > u\} \leq 1 - \Phi(u) + \frac{2\sigma_1(S)}{\sqrt{2\pi}}\varphi(u) + \frac{\sigma_2(S)\varphi(u)}{4\pi}\left[\sqrt{12\rho'' - 1}\varphi\left(\frac{u}{\sqrt{12\rho'' - 1}}\right) + \right.$$
$$\left. + \frac{\sigma_3(S)\varphi(u)}{(2\pi)^{\frac{3}{2}}}\left[u^2 - 1 + \frac{(8\rho'')^{\frac{3}{2}}\exp(-u^2.(24\rho'' - 2)^{-1})}{\sqrt{24\rho'' - 2}}\right]\right],$$

σ_1 caliper diameter ; σ_2 perimeter, σ_3 Lebesgue measure.

Examples

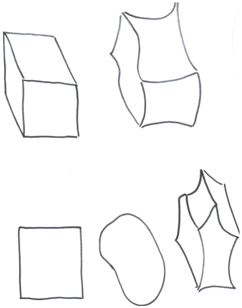
The record method $d = 2$ or 3

The maxima method

Second order

Processes defined on fractal sets

Consider a realization with $M > u$, then necessarily there exist a local maxima or a border maxima above U
Border maxima : local maxima in relative topology
If the consider sets that are union of manifolds of dimension 1 to d of this kind



In fact result are simpler (and stronger) in term of the density $p_M(x)$ of the maximum. Bound for the distribution are obtained by integration.

Theorem

$$p_M(x) \le \widehat{p}_M(x) := \frac{1}{2}[\overline{p}_M(x) + p_M^{EC}(x)] with$$

$$\overline{p}_M(x) := \int_S \mathrm{E}\big(|\det(X''(t))|/X(t) = x, X'(t) = 0\big) p_{X(t),X'_j(t)}(x,0)dt + \textit{boundary term}$$

and

$$p_M^{EC}(x) := (-1)^d \int_S \mathrm{E}\big(\det(X''(t))/X(t) = x, X'(t) = 0\big) p_{X(t),X'_j(t)}(x,0)dt + \textit{boundary}$$

Quantity $p_M^{EC}(x)$ is easy to compute using the work by Adler and properties of symmetry of the order 4 tensor of variance of X'' (under the conditional distribution))

Lemma

$$\mathrm{E}\Big(\det(X''(t))/X(t) = x, X'(t) = 0\Big) = \det(\Lambda) H_d(x)$$

where $H_d(x)$ is the dth Hermite polynomial and $\Lambda := \mathrm{Var}(X'(t))$

main advantage of Euler characteristic method lies in this result.

computation of $\overline{p}_m$

The key point is the following
If X is stationary and isotropic with covariance $\rho(\|t\|^2)$ normalized by $\mathrm{Var}(X(t)) = 1$ et $\mathrm{Var}(X'(t)) = Id$
Then under the condition $X(t) = x, X'(t) = 0$

$$X''(t) = \sqrt{8\rho"}G + \xi\sqrt{\rho" - \rho'^2}Id + xId$$

Where G is a GOE matrix (Gaussian Orthogonal Ensemble), and ξ a standard normal independent variable. We use recent result on the the characteristic polynomials of the GOE. Fyodorov(2004)

Theorem

Assume that the random field $\mathcal{X}$ is centered, Gaussian, stationary and isotrpic and is "regular" Let S have polyhedral shape. Then,

$$\overline{p}(x) = \varphi(x) \left\{ \sum_{t \in S_0} \widehat{\sigma}_0(t) + \sum_{j=1}^{d_0} \left[\left(\frac{|\rho'|}{\pi} \right)^{j/2} H_j(x) + R_j(x) \right] g_j \right\} \quad (1)$$

- $g_j = \int_{S_j} \widehat{\theta}_j(t) \sigma_j(dt)$, $\widehat{\sigma}_j(t)$ *is the normalized solid angle of the cone of the extended outward directions at t in the normal space with the convention $\sigma_d(t) = 1$.*
  For convex or other usual polyhedra $\widehat{\sigma}_j(t)$ is constant on faces of S_j,
- H_j *is the j th(probabilistic) Hermite polynomial.*

Theorem (continued)

- $R_j(x) = \left(\frac{2\rho''}{\pi|\rho'|}\right)^{\frac{j}{2}} \frac{\Gamma((j+1)/2}{\pi} \int_{-\infty}^{+\infty} T_j(v) \exp\left(-\frac{y^2}{2}\right) dy$

$$v := -(2)^{-1/2}\left((1-\gamma^2)^{1/2}y - \gamma x\right) \text{ with } \gamma := |\rho'|(\rho'')^{-1/2}, \qquad (2)$$

$$T_j(v) := \left[\sum_{k=0}^{j-1} \frac{H_k^2(v)}{2^k k!}\right] e^{-v^2/2} - \frac{H_j(v)}{2^j (j-1)!} I_{j-1}(v), \qquad (3)$$

$$I_n(v) = 2e^{-v^2/2} \sum_{k=0}^{[\frac{n-1}{2}]} 2^k \frac{(n-1)!!}{(n-1-2k)!!} H_{n-1-2k}(v) + \mathbb{I}_{\{n \text{ even}\}} 2^{\frac{n}{2}} (n-1)!! \sqrt{2\pi}(1-\Phi(x)) \qquad (4)$$

Second order study

Using an exact implicit formula

Theorem

Under conditions above + $\mathrm{Var}(X(t)) \equiv 1$ *Then*

$$\underline{\lim}_{x\to+\infty} \; -\frac{2}{x^2}\log\left[\widehat{p}_M(x) - p_M(x)\right] \geq 1 + \inf_{t\in S}\frac{1}{\sigma_t^2 + \overline{\lambda}(t)\kappa_t^2}$$

$$\sigma_t^2 := \sup_{s\in S\setminus\{t\}} \frac{\mathrm{Var}\big(X(s)\big|X(t), X'(t)\big)}{(1-r(s,t))^2}$$

and κ_t *is some geometrical characteristic et* $\Lambda_t = GEV(\Lambda(t))$

The right hand side is finite and > 1

Examples

The record method $d = 2$ or 3

The maxima method

Second order

Processes defined on fractal sets

In some applications we consider a Gaussian stationary process defined on a fractal set. For example the level set of some fractal function.
It is direct consequence of the results above that if $X(t)$ is normalized (centred, var =1) and with differentiable paths :

$$\mathbb{P}\{M > u\} \simeq u^{d-1}\phi(u)$$

for a parameter set S with integer dimension d.
What happens if the dimension d is fractal ? ? ?

The answer is positive if the set is Minkowsky measurable.
Let S^ϵ is the tube of size ϵ around S, we ask that

$$\lambda(S^\epsilon) \simeq C\epsilon^{n-d}.$$

λ is the Lebesgue measure, n the dimension of the ambient space, C is called the Minkowski content.
The proof is based on the fact that for a large level u and except with a negligible probability, in a neighborhood of the set S

- there is only one connected component above u.
- There is only one local maxima above u
- The connected component is almost a ball with center the maxima (and random radius r)
- Roughly speaking, the maximum on S is large than u if the maximum belongs to S^r.

Some references

Adler R.J. and Taylor J. E. *Random fields and geometry*. Springer.

Azaïs,J-M. and Wschebor M. *Level set and extrema of random processes and fields*, Wiley (2009).

Mercadier, C. (2006), Numerical Bounds for the Distribution of the Maximum of Some One- and Two-Parameter Gaussian Processes, *Adv. in Appl. Probab.* **38**, pp. 149–170.

Pham V. H., C. (2009), The upper bound for the tail of the distribution of some Gaussian fields. *in preparation*

THANK-YOU