

# Fluctuation Properties of Ideal $1/f$ Noise: from **Statistical Mechanics** to **Random Matrices**, **Riemann Zeta-Function** and **Burgers Turbulence**

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## **References:**

**Y V F, J-P Bouchaud** : J. Phys.A: Math.Theor 41 (2008), 372001

**Y V F, P Le Doussal , and A Rosso** : J. Stat. Mech. (2009) P10005 ;

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**Y V F, J.P. Keating** under preparation

## Definition of ideal 1/f noises:

Random signals such that **spectral power** associated with a given Fourier harmonic is **inversely proportional** to the frequency. Believed to be **frequently encountered** in Nature: voltage fluctuations, non-equilibrium phase transitions, spontaneous brain activity, etc.

- **Periodic version:** random **Fourier series** of the form

$$V(t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} [v_n e^{int} + \bar{v}_n e^{-int}] , \quad t \in [0, 2\pi)$$

where  $v_n, \bar{v}_n$  are **complex Gaussian i.i.d.** with zero mean and the variance  $\langle v_n \bar{v}_n \rangle = 1$ . It implies

$$\langle V(t_1)V(t_2) \rangle_V = 2 \sum_{n=1}^{\infty} \frac{1}{n} \cos n(t_1 - t_2) \equiv -2 \ln |2 \sin \frac{t_1 - t_2}{2}|, \quad t \neq t'$$

- **Aperiodic version:** similarly, random Gaussian **Fourier integral** defines a Gaussian process on the whole line  $-\infty < t < \infty$  by

$$V(t) = \int_0^{\infty} \frac{d\omega}{\sqrt{\omega}} [e^{i\omega t} v(\omega) + e^{-i\omega t} \bar{v}(\omega)], \quad \langle V(t_1)V(t_2) \rangle_V = -2 \ln |t_1 - t_2|$$

with  $\delta$ -correlated complex Gaussian  $v(\omega)$ . **The corresponding definitions are formal, as sums/integrals do not converge pointwise, and should be understood as random **generalized functions** (e.g. 1D "projections" of the **Gaussian Free Field**) or after a proper **regularization**.**

## A regularization:

Subdivide the interval  $t \in [0, 2\pi)$  by **finite number** of points  $t_k = \frac{2\pi}{M}k$  where  $k = 1, \dots, M < \infty$  and associate with each  $k$  Gaussian-distributed real variables  $V_k$  with **covariances**

$$\langle V_k V_m \rangle = -2 \ln |2 \sin \frac{t_k - t_m}{2}|, \quad \text{for } k \neq m$$

For the problem to be well-defined we have to choose the **variance** accordingly:

$$\langle V_k^2 \rangle = 2 \ln M + W, \quad \text{with any } W > 0$$

In the limit  $M \rightarrow \infty$  this is expected to approximate the  $2\pi$ -periodic **1/f noise** with the covariance  $\langle V(t_1)V(t_2) \rangle_V = -2 \ln |2 \sin \frac{t_1 - t_2}{2}|$ ,  $t \neq t' \in [0, 2\pi)$ .

- Our aim is to understand the statistics of **high/low** and **extreme** values of this **strongly correlated** sequence. The problem turns out to be intimately connected to the mechanism of **freezing transitions** in disordered systems theory (Random Energy Models, Dirac fermions in random magnetic field). It has also interesting relations to **Liouville Quantum Gravity**, random conformal weldings, to **multifractal** random measures in turbulence and mathematical finance, as well as to various aspects of the **Random Matrix Theory** and value distribution of the **Riemann zeta**-function along the critical line.

## Part I: mapping to Statistical Mechanics:

We interpret the sequence  $V_k$  for  $k = 1, \dots, M < \infty$  as a set of **random energies** and consider the associated equilibrium Statistical Mechanics by introducing the **temperature**  $T = \beta^{-1}$  and defining the **partition function**  $Z(\beta) = \sum_{i=1}^M e^{-\beta V_i}$ .

In this way we arrive to **1D generalization** of the **Derrida's Random Energy Model** to be studied in the thermodynamic limit  $M \rightarrow \infty$ . In particular, **minimal energy** can be extracted from the zero-temperature limit of the free energy as

$$V_{min} = \min(V_1, \dots, V_M) = \lim_{\beta \rightarrow \infty} f(\beta), \quad f(\beta) = -\beta^{-1} \log Z(\beta)$$

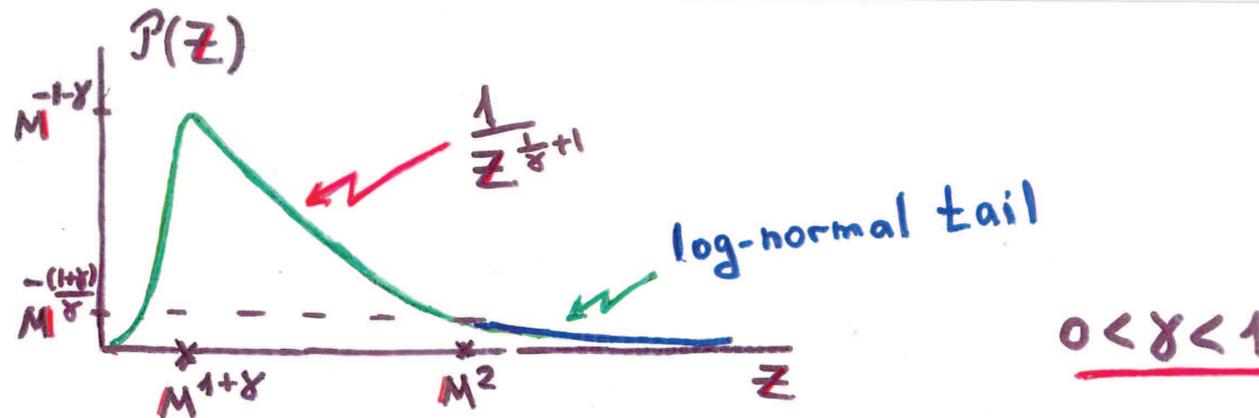
**Conclusion:** We need to know the statistics of the **free energy** to extract the **extreme value statistics** of the energy sequence.

**Observation:** The positive integer moments  $\langle Z^n(\beta) \rangle$ ,  $n = 1, 2, \dots$  of the partition function  $Z(\beta) = \sum_{i=1}^M e^{-\beta V_i}$  for the (regularized) periodic **1/f noise** sequence  $V_k$  in the **high-temperature** phase  $0 < \beta < 1$  turn out to be given in the thermodynamic limit  $M \gg 1$  by the **Dyson-Morris-Selberg** integral:

$$\langle Z^n(\beta) \rangle = \begin{cases} M^{1+\beta^2 n^2} O(1) & \text{for } n > 1/\beta^2 \\ M^{n(1+\beta^2)} \frac{\Gamma(1-n\beta^2)}{\Gamma^n(1-\beta^2)} & \text{for } 1 < n < 1/\beta^2 \end{cases}$$

We reconstruct the probability density  $\mathcal{P}(Z)$  from its moments.

**Outcome of the analysis:** The probability density  $\mathcal{P}(Z)$  of the partition function  $Z(\beta)$  in the high-temperature phase  $\gamma = \beta^2 < 1$  consists of two pieces:



The **"body"** of the distribution has a pronounced maximum at

$Z \sim Z_e = \frac{M^{1+\beta^2}}{\Gamma(1-\beta^2)} \ll M^2$ , and the powerlaw decay at  $Z_e \ll Z \ll M^2$ :

$$\mathcal{P}(Z) = \frac{1}{\beta^2} \frac{1}{Z} \left(\frac{Z_e}{Z}\right)^{\frac{1}{\beta^2}} e^{-\left(\frac{Z_e}{Z}\right)^{\frac{1}{\beta^2}}}, \quad Z \ll M^2$$

Limit  $\beta \rightarrow 0$  can be used to reproduce the Gumbel distribution of "roughness" by **Antal et al. '01**

At  $Z \gg M^2$  the above expression is replaced by a **lognormal tail**. Now we define  $z = Z/Z_e$  and consider the generating function

$$g_\beta(x) = \left\langle \exp(-e^{\beta x} z) \right\rangle_{M \gg 1}, \quad \beta = 1/T$$

## High-temperature "duality" and freezing scenario:

In the high-temperature phase the generating function  $g_\beta(x)$  can be found explicitly and turned out to satisfy a remarkable **duality relation**:

$$g_\beta(x) = \int_0^\infty dt \exp \left\{ -t - e^{\beta x} t^{-\beta^2} \right\} = g_{\frac{1}{\beta}}(x), \quad \beta < \beta_c = 1$$

This however does not allow to continue to  $\beta > \beta_c$  regime. Instead, the phase transition at  $\beta = \beta_c$  is **conjectured** to be described by the following **freezing scenario**:  $g_\beta(x)$  **freezes** to the **temperature independent** profile  $g_{\beta=\beta_c}(x)$  in the "glassy" phase  $T \leq T_c$ . The scenario is supported by

(i) a (one-loop) renormalization group arguments for the logarithmic models (**Carpentier, Le Doussal '01**) revealing an analogy to the **travelling wave analysis** of polymers on disordered trees (**Derrida, Spohn 1989**). In particular, the latter model shares with our model the REM-like **mean** free energy which in the limit  $M \gg 1$  behaves in the high-temperature phase  $\beta < 1$  as  $\langle f(\beta) \rangle \approx -\left(\beta + \frac{1}{\beta}\right) \log M$  and **"freezes"** to the critical value  $\langle f(\beta_c) \rangle \approx -2 \log M \equiv V_{min}$  for all temperatures below the transition.

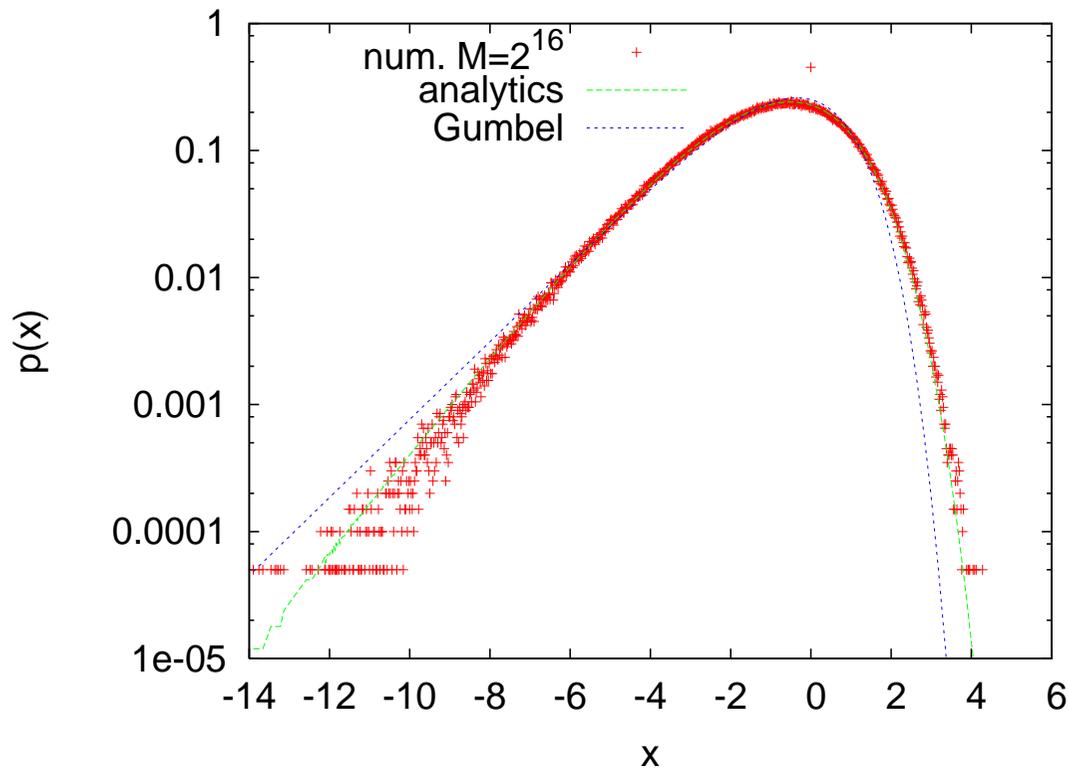
(ii) compatibility with **duality** which implies  $\partial_\beta g_\beta(x)|_{\beta=\beta_c^-} = 0$ , for all  $x$ , so that the "temperature flow" of this function vanishes at the critical point  $\beta = \beta_c = 1$ .

(iii) our numerics, also confirming the relation between the **freezing** and the **one-step replica symmetry breaking** mechanism operative in the "glassy" phase.

Assuming the **freezing scenario**, the **absolute minimum** of the random sequence is simply given by  $V_{min} = -\lim_{T \rightarrow 0} f = -2 \log M + c \log \log M + x$ , with unknown  $c$  (conjectured to be  $c = 3/2$ ) and the probability density of  $x$  related to the frozen profile  $g_{\beta_c}(x)$  by

$$p(x) = -g'_{\beta_c}(x) = -\frac{d}{dx} \left[ 2e^{x/2} K_1(2e^{x/2}) \right], \quad (1)$$

different from the **Gumbel law**  $p_{Gum}(x) = -\frac{d}{dx} \exp(-e^x)$ . Note the **tail**:  $p|_{x \rightarrow -\infty} \approx -xe^x$ .



**Distribution of extremes: we compare three distributions: (i) the histogram for ensemble of  $10^6$  realizations of the periodic  $1/f$  model sampled at  $M = 2^{16}$  equispaced points, (ii) the analytical prediction (1), and (iii) the Gumbel distribution for the mean & variance given by (1)**

## Multifractal structure of low/high values of the ideal 1/f noises:

Remembering that asymptotically  $V_{min} = -2 \log M$ , let us call the value  $V_i$  “**x–low**” if  $V_i < 2x \log M$ , for some  $x \in (-1, 0)$ . Our next goal is to count the **total number**  $\mathcal{N}_M(x)$  of **x–low** points in the **1/f sequence**. A direct calculation for periodic case shows that the number of **x–low** points is **multifractal**:

$$\mathcal{N}_{M \gg 1}(x) \sim \mathcal{N}_e = \frac{M^{1-x^2}}{2|x|\sqrt{\pi \log M} \Gamma(1-x^2)}, \quad 0 < |x| < 1.$$

The ratio  $n = \frac{\mathcal{N}_M(x)}{\mathcal{N}_e}$  is randomly fluctuating from one realization of the process to the other according to the probability density ( **YF, Le Doussal, & Rosso, in progress**):

$$\mathcal{P}(n) \approx \frac{1}{x^2} n^{-\frac{1}{x^2}-1} \exp\left(-n^{-\frac{1}{x^2}}\right), \quad n \ll n_{max} = \frac{M}{\mathcal{N}_e} \sim M^{x^2}, \quad 0 < |x| < 1$$

and vanishing very fast when  $n \sim n_{max}$ .

**Note:**  $M \gg 1$  limit of the discrete regularization of the **aperiodic** version of the **1/f** noise can be considered by similar techniques **Y F, Le Doussal, Rosso, 2009**. Qualitatively, the two cases share all the major features of the extreme and high value statistics (in particular, **multifractality** and the same **power-law tail** for the distribution of the number of points above given high level). Details of the distributions are however different.

## Part II: From 1/f noises to Random Matrices and Riemann zeta-function:

let  $U_N$  be a  $N \times N$  **unitary matrix**, chosen at random from the unitary group  $\mathcal{U}(N)$ . Introduce its **characteristic polynomial**  $p_N(\theta) = \det(1 - U_N e^{-i\theta})$  and define the following objects which are formal analogues of the **partition function**

$$\mathcal{Z}_N(\beta; L) = \frac{N}{2\pi} \int_0^{2\pi} |p_N(\theta)|^{2\beta} d\theta \equiv \frac{N}{2\pi} \int_0^{2\pi} e^{-\beta V_N(\theta)} d\theta, \quad \beta > 0$$

where  $V_N(\theta) = -2 \log |p_N(\theta)|$ . We observe that it is possible to evaluate the  $N \gg 1$  limit of the positive integer moments

$$\mathbb{E} \{ \mathcal{Z}_N^n(\beta; L) \} = N^n \int_0^{2\pi} \dots \int_0^{2\pi} \mathbb{E} \{ |p_N(\theta_1)|^{2\beta} \dots |p_N(\theta_n)|^{2\beta} \} \prod_{j=1}^n \frac{d\theta_j}{2\pi},$$

where the expectation  $\mathbb{E}\{\dots\}$  stand for the  $\mathcal{U}(N)$  group average. Indeed, the expectation value in the integrand is a **Toeplitz determinant** whose asymptotic  $N \gg 1$  behaviour is known due to **Fisher-Hartwig** and **Widom**:

$$\mathbb{E} \{ |p_N(\theta_1)|^{2\beta} \dots |p_N(\theta_n)|^{2\beta} \} \sim \left[ N^{\beta^2} \frac{G^2(1+\beta)}{G(1+2\beta)} \right]^n \prod_{r < s}^n |e^{i\theta_r} - e^{i\theta_s}|^{-2\beta^2}$$

We then see that the  $\theta$ -integral is again of the same **Dyson-Morris-Selberg** type that we have studied before.

**Conclusion:** Log-Mod of the characteristic polynomial of CUE matrices is just a **different regularization** of the **same** periodic **1/f noise**!

In the last decade, following **Keating & Snaith 2001**, it became a **well-accepted paradigm** that

many properties of the **Riemann zeta-function**  $\zeta(s)$  along the **critical line**  $s = \frac{1}{2} + it$ ,  $t \in \mathbb{R}$  can be successfully understood by comparing them to analogous properties of **characteristic polynomials** of random matrices of large size  $N \sim \log t$ . The idea is inspired by random-matrix properties of zeta-function zeroes (**Montgomery (1973); Odlyzko, ...** ). To this end, define for a fixed real  $t$  the function

$$V_t^{(\zeta)}(x) = \log |\zeta(\frac{1}{2} + i(t+x))| = \mathbf{Re} \log \zeta(\frac{1}{2} + i(t+x))$$

For large  $t \rightarrow \infty$  the function  $V_t^{(\zeta)}(x)$  actually mimics a **Gaussian random function** of variable  $x$  of mean zero and variance  $\frac{1}{2} \log \log t$  (**Selberg**, see also **Hughes, Keating, O'Connell**). Moreover, a simple consideration which uses the **Euler product formula** for Riemann zeta and the **probabilistic** properties of primes given by the **Prime Number Theorem** allows one to show that the small- $x$  behaviour of the covariance high up the critical line:

$$\langle V_t^{(\zeta)}(x_1) V_t^{(\zeta)}(x_2) \rangle \approx \begin{cases} -\frac{1}{2} \log |x_1 - x_2|, & \mathbf{for} \quad \frac{1}{\log t} \ll |x_1 - x_2| \ll 1 \\ \frac{1}{2} \log \log t, & \mathbf{for} \quad |x_1 - x_2| \ll \frac{1}{\log t} \end{cases}$$

with the averaging going over an interval  $[t - h/2, t + h/2]$  such that  $\frac{1}{\log t} \ll h \ll t$ .

**Message:** **locally** the log of Riemann zeta-function resembles aperiodic **1/f noise**. One can use it to cast new light on statistics of **high values** of the Riemann zeta along the critical line (**Y V F & JP Keating**, work in progress.)

### Part III: Decaying Burgers Turbulence:

- The problem of analysis of solutions of the (unforced) Burgers equation

$$\partial_t \mathbf{v} + (\mathbf{v} \nabla) \mathbf{v} = \nu \nabla^2 \mathbf{v}, \quad \mathbf{v}(\mathbf{x}, t = 0) = -\nabla \Psi_0(\mathbf{x}), \quad \nu > 0$$

with random initial condition, usually assumed to be Gaussian and specified in terms of the two-point correlation function  $\overline{\mathbf{v}(\mathbf{x}, 0) \mathbf{v}(\mathbf{x}', 0)}$ , or alternatively  $\overline{\Psi_0(\mathbf{x}) \Psi_0(\mathbf{x}')}$ . General reference: **Bec & Khanin** Physics Reports **447** (2007), 1

- The problem appears as an important reference model not only in fluid dynamics, but also in such diverse physical contexts as statistical mechanics of systems with quenched disorder (**Balents, Bouchaud, Mezard '95; Le Doussal '08**), and formation of large scale structures in cosmology (**Gurbatov, Saichev, Shandarin '89; Vergassola, Dubrulle, Frish, Noullez '94**). In particular, the cosmological applications stimulated interest in **dB**T for **vanishing viscosity**  $\nu \rightarrow 0$  and **scale-free** power-law random initial conditions:

$$\overline{\mathbf{v}(\mathbf{x}, 0) \mathbf{v}(\mathbf{x}', 0)} \sim |\mathbf{x} - \mathbf{x}'|^{-n-1} \quad \text{at large distances}$$

## Cole-Hopf solution, mapping to Statistical Mechanics:

Solution to decaying Burgers turbulence for a given initial potential  $\Psi_0(\mathbf{x})$  can be written explicitly as

$$\mathbf{v}(\mathbf{x}, t) = -2\nu \nabla_x \ln Z, \quad Z(\mathbf{x}, t) = \int e^{-\frac{1}{2\nu} \mathcal{H}_x(\mathbf{y})} \frac{d\mathbf{y}}{(4\pi\nu t)^{d/2}}$$

in terms of the **"effective potential"**

$$\mathcal{H}_x(\mathbf{y}) = \Psi_0(\mathbf{y}) + \frac{1}{2t}(\mathbf{x} - \mathbf{y})^2$$

Such  $Z(\mathbf{x}, t)$  can be interpreted as the **partition function** of a particle equilibrated in the energy potential  $\mathcal{H}_x(\mathbf{y})$  at an **effective temperature**  $T = 2\nu$ , so that **inviscid limit = zero temperature**).

**Observation:** for  $n = 1$  the initial velocity decays as  $\overline{v(x, 0)v(x', 0)} \sim |x - x'|^{-2}$  which implies the potential  $\Psi_0(\mathbf{x})$  is a version of **1/f noise**:

$$\overline{\Psi_0(x)\Psi_0(x')} = -2 \ln [ |x - x'|/L ], \quad \epsilon < |x - x'| < L$$

where  $L \gg 1$  and  $\epsilon \ll 1$  are the infrared and ultraviolet cutoff scales. We therefore may anticipate a kind of **freezing transition** inducing changes in the shape of the p.d.f. for velocity  $v(x, t) = -2\nu \partial_x \ln Z(x, t)$  at finite **critical viscosity**  $\nu_c (\equiv T_c/2) > 0$ .

**Note:** it is commonly accepted in the literature that that the velocity  $v(x, t)$  remains **always Gaussian-distributed** for any viscosity, down to the inviscid limit  $\nu = 0$ .

## Statistical mechanics in random potential and Burgers velocity:

In the language of statistical mechanics with  $T = 2\nu$  the velocity is given by

$$v = -\frac{1}{t} \langle y \rangle_T \quad \text{where} \quad \langle O \rangle_T = \frac{1}{Z(x,t)} \int \frac{dy}{\sqrt{2\pi T t}} O(y) e^{-\mathcal{H}_0(y)/T}$$

where  $\mathcal{H}_0(y) = \Psi_0(y) + \frac{y^2}{2t}$  is the random energy function.

To understand better **thermodynamics** of our system and the nature of the anticipated **freezing transition** it turns out to be instructive to consider also a different object:

$$\mathcal{P}_Y(Y) = \overline{\langle \delta(Y - y) \rangle_T} = \overline{\frac{1}{Z} e^{-\mathcal{H}_0(Y)/T}}$$

interpreted as the averaged **Boltzmann-Gibbs** probability of the coordinate of a particle **equilibrated** at a given temperature  $T$  in the random energy landscape  $\mathcal{H}_0(Y)$ . At  $T \rightarrow 0$  the thermal average is obviously dominated by the **deepest minimum** of the landscape whose position  $Y_{\min}$  fluctuates from one realization of disorder to the other. This mechanism immediately implies for velocity p.d.f. in **zero viscosity** ( $= T \rightarrow 0$ ) limit the relation

$$\mathcal{P}(v)|_{T=0} = t\mathcal{P}_Y(vt)|_{T=0}$$

## Statistical mechanics in random potential and $\lambda$ -Hermite ensemble:

The disorder averaging procedure for  $\mathcal{P}_Y(Y)$  can be performed via the standard **replica trick** after representing  $Z^{-1} = Z^{n-1}|_{n \rightarrow 0}$  and using the Gaussian nature of the random potential  $\Psi_0(y)$  by employing  $\overline{\Psi_0(y)\Psi_0(y')} = -2 \ln [|y - y'|/L]$ . One finds

$$\mathcal{P}_Y(Y) = \lim_{n \rightarrow 0} \left\langle \frac{1}{n} \sum_{j=1}^n \delta \left( Y - z_j \sqrt{Tt} \right) \right\rangle_{n, -\gamma}$$

where  $\gamma = 1/T^2 > 0$  and we have defined for  $1 \leq n < 1/\gamma$

$$\langle \dots \rangle_{n, \lambda} = \frac{1}{S_n(\lambda)} \int_{-\infty}^{\infty} (\dots) \prod_{i < j} |z_i - z_j|^{2\lambda} \prod_{j=1}^n \frac{dz_j}{\sqrt{2\pi}} e^{-\frac{z_j^2}{2}},$$

with  $S_n(\lambda) = \prod_{j=1}^{j=n} [\Gamma(1 + j\lambda)/\Gamma(1 + \lambda)]$  being the famous **Selberg** integral. For finite integer  $n \geq 1$  and  $\lambda > 0$  the above expression is nothing else but the mean density of the so-called  $\lambda$ -**Hermite** ensemble of  $n \times n$  random matrices introduced by **Dumitriu & Edelman** '02 .

Note that the corresponding random matrix-like integrals are still **convergent** for  $\lambda = -\gamma$  as long as  $0 < \gamma < 1$ . The **replica limit** implies  $n \rightarrow 0$ .

## Statistical mechanics in random potential and $\lambda$ -Hermite ensemble:

Although a closed-form expression for the eigenvalue density for  $\lambda$ -Hermite ensemble does not seem to be available, one can use analytic tools developed by **Dumitriu & Edelman** '02,'06 and find **a few lower moments** of that density explicitly. Performing the analytical continuation  $n \rightarrow 0$  and  $\lambda \rightarrow -\gamma$  we obtained the lower nonvanishing moments  $M_{2q} = \int \mathcal{P}_Y(Y) Y^{2q} dY$  up to  $2q = 16$ . We present below the corresponding **cumulants**  $C_{2q}$ :

$$C_2 = t (T + T^{-1}), \quad C_4 = -t^2, \quad C_6 = 2t^3 (T + T^{-1})$$

$$C_8 = -t^4 [26 + 6 (T^2 + T^{-2})], \quad C_{10} = t^5 [300 (T + T^{-1}) + 24 (T^3 + T^{-3})]$$

and similar but longer expressions for  $C_{2q}$ ,  $q = 6, 7, 8$ .

The main feature apparent from the above (and proved in full generality) is that **all the cumulants** (and hence the whole function  $\mathcal{P}_Y(Y)$ ) are **invariant** with respect to the **duality transformation**  $T \rightarrow 1/T$ . Employing the **freezing conjecture** we thus predict that **the whole Gibbs-Boltzmann probability density**  $\mathcal{P}_Y(Y)$  **freezes** at the critical point  $T = T_c = 1$  providing a vivid picture of what freezing entails.

## Freezing scenario vs. numerics for zero viscosity velocity moments:

If this scenario were correct, the values of the above cumulants evaluated at  $T = 1$  should immediately provide, in view of the discussed zero-temperature correspondence, the **cumulants of the velocity p.d.f.** in **zero viscosity** limit:

$$\overline{v^2}|_{\nu=0} = \frac{2}{t}, \quad \overline{v^4}^c = \left[ \overline{v^4} - 3\overline{v^2}^2 \right] |_{\nu=0} = -\frac{1}{t^2}, \text{ etc.}$$

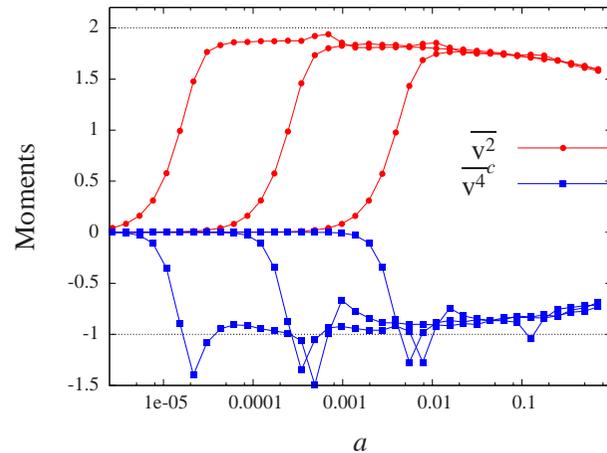


Figure 1: Numerical evaluation of  $\overline{v^2}$  and  $\overline{v^4}^c$  in the inviscid limit  $\nu = 0$  for discretized Burgers equation (number of points  $M = 2^{10}, 2^{14}, 2^{18}$ ) with periodic version of the logarithmically correlated potential (averaged over  $10^6$  samples) against the theoretical prediction at  $t = 1$ .

In fact, we can show that the velocity is **non-Gaussian** everywhere in the **low-viscosity** phase  $\nu < \nu_c = 1/2$ . Above the critical viscosity Gaussianity is restored.

## Summary:

**I.** Using the methods of statistical mechanics of disordered systems we studied the statistics of **minima/maxima** of the Gaussian **1/f noise**, both periodic and aperiodic. The distributions are manifestly **non-Gumbel** and show **universal backward tail**  $p|_{x \rightarrow -\infty} \approx -xe^x$ . This is heavily based on the conjectured **freezing scenario**, supported by numerics, high-temperature duality, REM-like replica-symmetry breaking, and renormalization-group arguments, but still lacking rigorous justification. We have also predicted a strongly-fluctuating **multifractal pattern** in the powerlaw-distributed **number of high/low points** in the **1/f** signals.

**II.** We reveal strong links between the **1/f noise** and Log-mod of **characteristic polynomials** of random matrices, and of the **Riemann** zeta-function along the critical line. This allows to put forward new conjectures about the statistics of high and extreme values of the latter objects.

**III.** Combining the methods of statistical mechanics with insights from the random matrix theory we reveal a phase transition with decreasing viscosity  $\nu$  at finite  $\nu = \nu_c > 0$  in one-dimensional decaying Burgers turbulence with a power-law correlated random profile of initial velocities  $\overline{v(x, 0)v(x', 0)} \sim |x - x'|^{-2}$ . The low-viscosity phase exhibits **non-Gaussian one-point probability density of velocities**, reflecting a **spontaneous one step replica symmetry breaking (RSB)** in the associated statistical mechanics problem. We obtain the low orders cumulants analytically which favourably agree with numerical simulations.