KPZ interface and directed polymer via the replica method

P. Le Doussal

with : Pasquale Calabrese (Univ. Pise)

Alberto Rosso (LPTMS Orsay)

P. Calabrese, P. Le Doussal, A. Rosso EPL 90 20002 (2010)

Outline

- directed polymer, discrete and continuum, KPZ equation

- quantum mechanics + replica , high T, Lieb Liniger model

- Bethe Ansatz

Outline

- directed polymer, discrete and continuum, KPZ equation

- quantum mechanics + replica , high T, Lieb Liniger model

- Bethe Ansatz

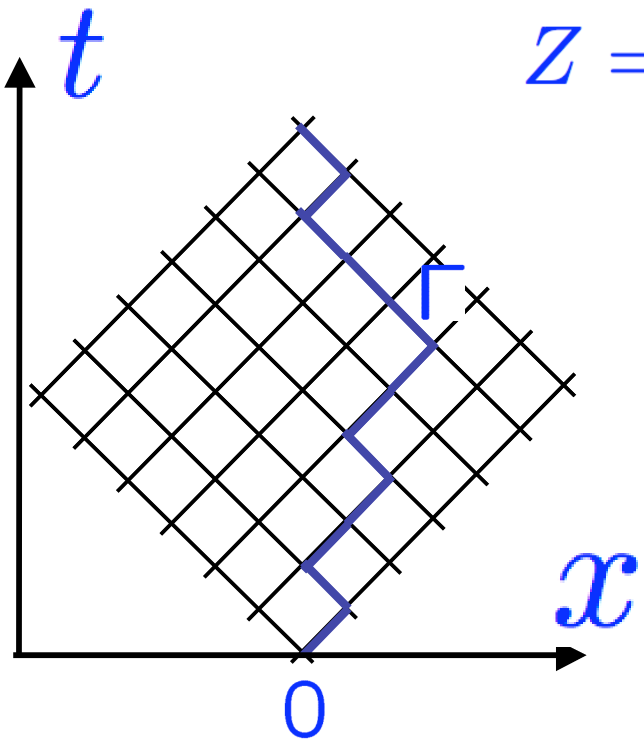
- moments of partition sum Z^n for DP with fixed endpoint
= KPZ with droplet initial condition + numerical checks

- generating function of Z^n can be expressed
as a Fredholm determinant, obtain distrib. free energy

Outline

- directed polymer, discrete and continuum, KPZ equation
- quantum mechanics + replica , high T, Lieb Liniger model
- Bethe Ansatz
- moments of partition sum Z^n for DP with fixed endpoint
  = KPZ with droplet initial condition + numerical checks
- generating function of Z^n can be expressed
  as a Fredholm determinant, obtain distrib. free energy
- large time limit recovers Tracy Widom GUE
- DP 1 free endpoint=KPZ flat init. cond.
  Fredholm Pfaffian and TW for GOE

directed polymer: 1) lattice model



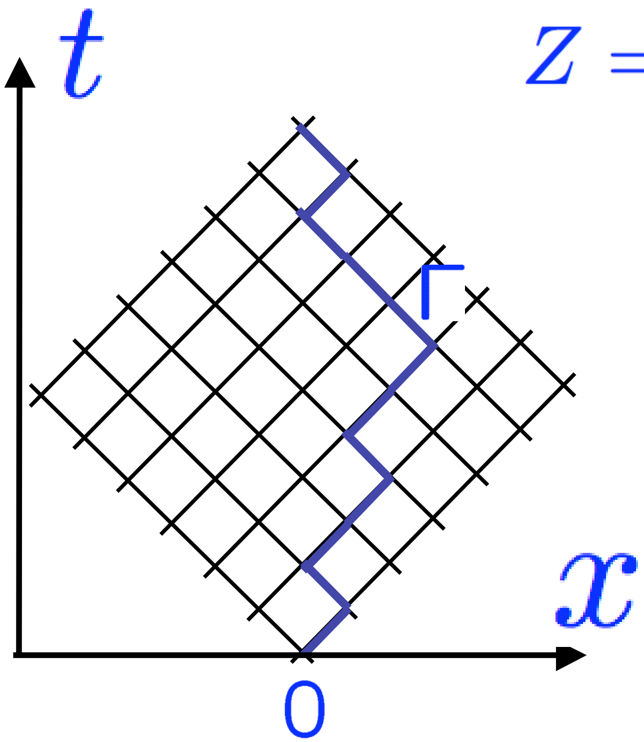
$$Z = \sum_{paths\,\Gamma} e^{-E_\Gamma/T}$$

V_i random variables

$$E_\Gamma = \sum_{i\in\Gamma} V_i$$

$$F = -T \ln Z$$

directed polymer: 1) lattice model



$$Z = \sum_{paths\Gamma} e^{-E_\Gamma/T}$$

V_i random variables

$$E_\Gamma = \sum_{i\in\Gamma} V_i$$

$$F = -T\ln Z$$

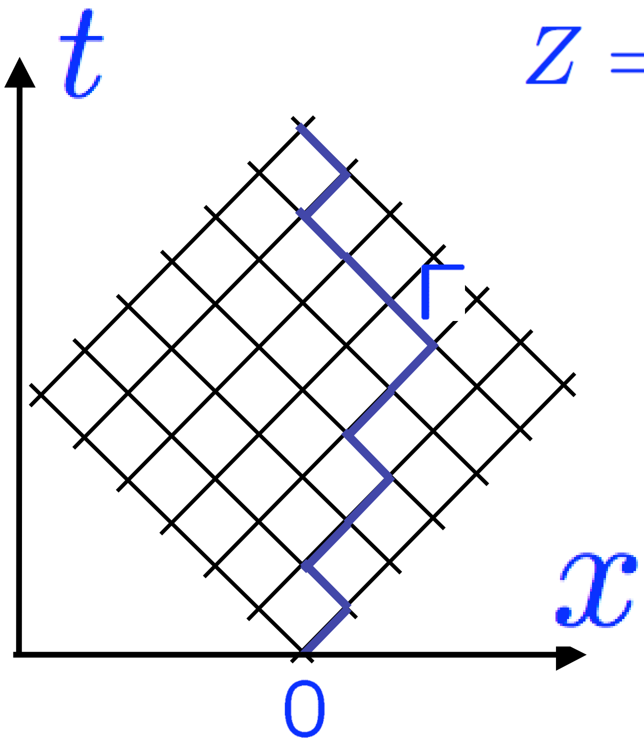
$$F_{T=0} = E_0 = \min_\Gamma E_\Gamma$$

$$\overline{E_0} = e_0 t \qquad (\overline{E_0^2}^c)^{1/2} = \sigma\omega t^\theta \qquad \overline{x(t)^2} \sim t^{2\zeta}$$

Johansson 2000
T=0 proof

$$\theta = \frac{1}{3} \qquad \zeta = \frac{2}{3}$$

directed polymer: 1) lattice model



$$Z = \sum_{paths\,\Gamma} e^{-E_\Gamma/T}$$

V_i random variables

$$E_\Gamma = \sum_{i\in\Gamma} V_i$$

$$F = -T \ln Z$$

$$F_{T=0} = E_0 = \min_\Gamma E_\Gamma$$

$$\overline{E_0} = e_0 t \qquad (\overline{E_0^2}^c)^{1/2} = \sigma\omega t^\theta \qquad \overline{x(t)^2} \sim t^{2\zeta}$$

$$\tilde{T} = T/\Delta F \sim t^{-1/3}$$

Johansson 2000
T=0 proof

$$\theta = \frac{1}{3} \qquad \zeta = \frac{2}{3}$$

directed polymer: 2) continuum model

$$Z(x,y,t) = \int_{x(0)=x}^{x(t)=y} Dx e^{-\frac{1}{T}\int_0^t d\tau [\frac{\kappa}{2}(\frac{dx}{d\tau})^2 + V(x(\tau),\tau)]}$$

$$\overline{V(x,t)V(x',t)} = \delta(t-t')R(x-x')$$

Feynman Kac

$$\partial_t Z = \frac{T}{2\kappa}\partial_x^2 Z - \frac{V(x,t)}{T} Z$$

$$Z(x,y,t=0) = \delta(x-y)$$

directed polymer: 2) continuum model

$$Z(x,y,t) = \int_{x(0)=x}^{x(t)=y} Dx e^{-\frac{1}{T}\int_0^t d\tau [\frac{\kappa}{2}(\frac{dx}{d\tau})^2 + V(x(\tau),\tau)]}$$

$$\overline{V(x,t)V(x',t)} = \delta(t-t')R(x-x')$$

can one take ?

$$R(x) \to \delta(x)$$

r_f

Feynman Kac

$$\partial_t Z = \frac{T}{2\kappa}\partial_x^2 Z - \frac{V(x,t)}{T} Z$$

directed polymer: 2) continuum model

$$Z(x,y,t) = \int_{x(0)=x}^{x(t)=y} Dx e^{-\frac{1}{T}\int_0^t d\tau [\frac{\kappa}{2}(\frac{dx}{d\tau})^2 + V(x(\tau),\tau)]}$$

$$\overline{V(x,t)V(x',t)} = \delta(t-t')R(x-x')$$

can one take ?

$$R(x) \to \delta(x)$$

Feynman Kac

$$\partial_t Z = \frac{T}{2\kappa}\partial_x^2 Z - \frac{V(x,t)}{T} Z$$

$$\nu = \frac{T}{2\kappa}, \lambda_0 \eta(x,t) = \frac{-V(x,t)}{\kappa}$$

Cole Hopf $\lambda_0 h(x,t) = T \ln Z(x,t)$

KPZ

$$\partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2}(\partial_x h)^2 + \eta(x,t)$$

directed polymer: 2) continuum model

$$Z(x,y,t) = \int_{x(0)=x}^{x(t)=y} Dx e^{-\frac{1}{T}\int_0^t d\tau [\frac{\kappa}{2}(\frac{dx}{d\tau})^2 + V(x(\tau),\tau)]}$$

$$\overline{V(x,t)V(x',t)} = \delta(t-t')R(x-x')$$

can one take ?

$$R(x) \to \delta(x)$$

Feynman Kac

$$\partial_t Z = \frac{T}{2\kappa}\partial_x^2 Z - \frac{V(x,t)}{T} Z$$

$$\nu = \frac{T}{2\kappa}, \quad \lambda_0 \eta(x,t) = \frac{-V(x,t)}{\kappa}$$

Cole Hopf

$$\lambda_0 h(x,t) = T \ln Z(x,t)$$

KPZ

$$\partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2}(\partial_x h)^2 + \eta(x,t)$$

STS symmetry

$$\theta = 2\zeta - 1$$

if white noise

$$\overline{\eta(x,t)\eta(x',t')} = D\delta(t-t')\delta(x-x')$$

$$P[\{h(x)\}] \sim e^{-\frac{\nu}{2D}\int dx h'(x)^2}$$

$$h \sim x^{1/2} \sim x^{\frac{\theta}{\zeta}}$$

$$\zeta = 2\theta = 2/3$$

Quantum mechanics and Replica..

$$\mathcal{Z}_n := \overline{Z(x_1, y_1, t)..Z(x_n, y_n, t)} = \langle x_1, ..x_n | e^{-tH_n^{rep}} | y_1, ..y_n \rangle$$

$$\partial_t \mathcal{Z}_n = -H_n^{rep} \mathcal{Z}_n$$

$$H_n^{rep} = -\frac{T}{2\kappa} \sum_{i=1}^{n} \partial_{x_i}^2 - \frac{1}{2T^2} \sum_{ij} R(x_i - x_j)$$

Quantum mechanics and Replica..

$$\mathcal{Z}_n := \overline{Z(x_1, y_1, t)..Z(x_n, y_n, t)} = \langle x_1, ..x_n | e^{-tH_n^{rep}} | y_1, ..y_n \rangle$$

$$\partial_t \mathcal{Z}_n = -H_n^{rep} \mathcal{Z}_n$$

$$H_n^{rep} = -\frac{T}{2\kappa} \sum_{i=1}^{n} \partial_{x_i}^2 - \frac{1}{2T^2} \sum_{ij} R(x_i - x_j)$$

$$x = T^3 \kappa^{-1} \tilde{x} \quad , \quad t = 2T^5 \kappa^{-1} \tilde{t}$$

high T limit:

$$\tilde{R}(z) \to 2\bar{c}\delta(z)$$

$$\bar{c} = \int du R(u)$$

$$T^3 (\bar{c}\kappa)^{-1} \gg r_f$$

Quantum mechanics and Replica..

$$\mathcal{Z}_n := \overline{Z(x_1, y_1, t)..Z(x_n, y_n, t)} = \langle x_1, ..x_n | e^{-tH_n^{rep}} | y_1, ..y_n \rangle$$

$$\partial_t \mathcal{Z}_n = -H_n^{rep} \mathcal{Z}_n$$

$$H_n^{rep} = -\frac{T}{2\kappa} \sum_{i=1}^{n} \partial_{x_i}^2 - \frac{1}{2T^2} \sum_{ij} R(x_i - x_j)$$

high T limit:

$$x = T^3 \kappa^{-1} \tilde{x} \quad , \quad t = 2T^5 \kappa^{-1} \tilde{t}$$

$$\tilde{R}(z) \to 2\bar{c}\delta(z)$$

$$\bar{c} = \int du R(u)$$

$$T^3 (\bar{c}\kappa)^{-1} \gg r_f$$

drop the tilde..

$$H_{LL} = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{1 \le i < j \le n} \delta(x_i - x_j) \qquad c = -\bar{c}$$

Attractive Lieb-Lineger (LL) model (1963)

Quantum mechanics and Replica..

$$\mathcal{Z}_n := \overline{Z(x_1,y_1,t)..Z(x_n,y_n,t)} = \langle x_1,..x_n|e^{-tH_n^{rep}}|y_1,..y_n\rangle$$

$$\partial_t \mathcal{Z}_n = -H_n^{rep} \mathcal{Z}_n$$

$$H_n^{rep} = -\frac{T}{2\kappa}\sum_{i=1}^{n}\partial_{x_i}^2 - \frac{1}{2T^2}\sum_{ij} R(x_i - x_j)$$

high T limit:

$$x = T^3\kappa^{-1}\tilde{x} \quad , \quad t = 2T^5\kappa^{-1}\tilde{t}$$

$$\tilde{R}(z) \to 2\bar{c}\delta(z)$$

$$\bar{c} = \int du R(u)$$

$$T^3(\bar{c}\kappa)^{-1} \gg r_f$$

drop the tilde..

$$H_{LL} = -\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2} + 2c\sum_{1\le i<j\le n}\delta(x_i - x_j) \qquad c = -\bar{c}$$

bosons or fermions?

Attractive Lieb-Lineger (LL) model (1963)

Bethe ansatz: ground state

Kardar 87

n bosons+attraction = bound state

$$\psi_0(x_1, ..x_n) \sim \exp(-\frac{\bar{c}}{2} \sum_{i<j} |x_i - x_j|) \qquad E_0(n) = -\frac{\bar{c}^2}{12} n(n^2-1)$$

$$\overline{Z(x_1,0,t)..Z(x_n,0,t)} \approx_{t\to\infty} \psi_0(x_1, ..x_n) e^{-tE_0(n)}$$

$$\overline{Z^n} = \overline{e^{n \ln Z}} = e^{\sum_p \frac{1}{p!} n^p \overline{(\ln Z)^p}^c} \sim e^{\frac{\bar{c}^2}{12} n^3 t} \qquad \overline{(\ln Z)^3}^c \sim t \qquad \overline{(\ln Z)^2}^c \sim 0?$$

$$\sim e^{\frac{\bar{c}^2}{12} n^3 t + O(n^2 t^{2/3})}$$

Bethe ansatz: ground state

Kardar 87

n bosons+attraction = bound state

$$\psi_0(x_1, ..x_n) \sim \exp(-\frac{\bar{c}}{2} \sum_{i<j} |x_i - x_j|) \qquad E_0(n) = -\frac{\bar{c}^2}{12} n(n^2 - 1)$$

$$\overline{Z(x_1, 0, t)..Z(x_n, 0, t)} \approx_{t \to \infty} \psi_0(x_1, ..x_n) e^{-tE_0(n)}$$

$$\overline{Z^n} = \overline{e^{n \ln Z}} = e^{\sum_p \frac{1}{p!} n^p \overline{(\ln Z)^p}^c} \sim e^{\frac{\bar{c}^2}{12} n^3 t}$$

$$F = -\ln Z = \bar{F} + \lambda f \qquad \lambda = (\frac{\bar{c}^2}{4} t)^{1/3}$$

information about the tail of FE distribution

$$P(f) \sim_{f \to -\infty} \exp(-\frac{2}{3}(-f)^{3/2})$$

$$\overline{Z^n} = \int df e^{-n\lambda f - \frac{2}{3}(-f)^{3/2}} \sim e^{\frac{1}{3}\lambda^3 n^3}$$

FE distribution on a cylinder

Brunet Derrida (2000)

cylinder x+L = x

$$E(n,L) = -\lim_{t\to+\infty} \frac{1}{t} \frac{\overline{Z^n(x,t)}}{\overline{Z(x,t)}^n}$$

- Kardar $L = +\infty$

violates $\frac{\partial^2}{\partial n^2} E(n,L) \leq 0$

cannot be continued in n

- ground state on cylinder $E(n,L) = -\frac{1}{L^{3/2}} G(-nL^{1/2})$

$\sim n^3 \qquad nL \gg 1$

large deviation of FE distribution on cylinder

Q: distribution of free energy ln Z? <=> distribution of h(x,t) in KPZ
DP of finite length t

$$Z(x,t) = e^{\frac{\lambda_0}{2\nu} h(x,t)}$$

Here= CONTINUUM model (DP or KPZ) = BA + sum over all excited states

fixed t , hence $L = +\infty$ is ok

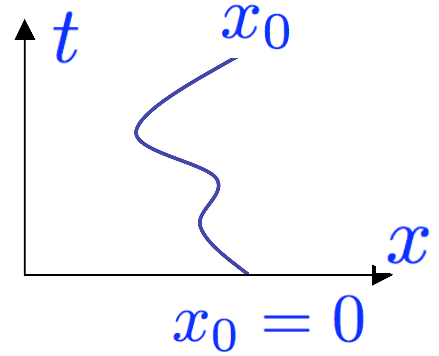
Q: distribution of free energy ln Z? <=> distribution of h(x,t) in KPZ
DP of finite length t

$$Z(x,t) = e^{\frac{\lambda_0}{2\nu} h(x,t)}$$

Here= CONTINUUM model (DP or KPZ) = BA + sum over all excited states

fixed t , hence $L = +\infty$ is ok

1) DP fixed endpoints



Johansson (2000) T=0

$$E_0 = e_0 t + \sigma \omega t^{1/3} \qquad P(V=q) \sim p^q$$

$$Prob(\omega > -s) = F_2(s)$$

Tracy Widom= largest eigenvalue of GUE

KPZ=narrow wedge, droplet initial condition

$$h(x,t=0) = -w|x| \qquad w \to \infty$$

Q: distribution of free energy ln Z? <=> distribution of h(x,t) in KPZ
DP of finite length t

$$Z(x,t) = e^{\frac{\lambda_0}{2\nu} h(x,t)}$$

Here= CONTINUUM model (DP or KPZ) = BA + sum over all excited states

fixed t , hence $L = +\infty$ is ok

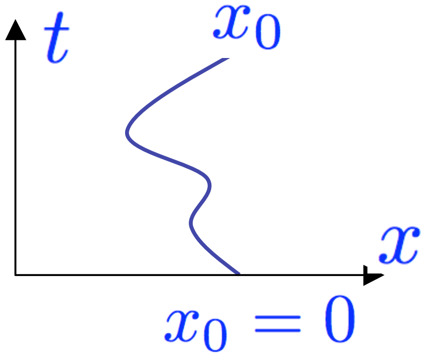
1) DP fixed endpoints

Johansson (2000) T=0

$$E_0 = e_0 t + \sigma \omega t^{1/3} \qquad P(V=q) \sim p^q$$

$$Prob(\omega > -s) = F_2(s) \qquad t \to +\infty$$

Tracy Widom= largest eigenvalue of GUE



KPZ=narrow wedge, droplet initial condition

$$h(x,t=0) = -w|x| \qquad w \to \infty$$

2) DP one fixed one free endpoint

$$e^{\frac{\lambda_0}{2\nu} h(x,t)} = \int dy Z(x,t|y,0) e^{\frac{\lambda_0}{2\nu} h(y,t=0)}$$

KPZ = flat initial condition $w \to 0$

$$h(x,t=0) = 0$$

PNG model (Spohn, Ferrari,..)

$$t \to +\infty \quad F_1(s)$$

- Continuum DP fixed endpoint/KPZ Narrow wedge

1) BA + replica

- P. Calabrese, P. Le Doussal, A. Rosso EPL 90 20002 (2010)
- V. Dotsenko, EPL 90 20003 (2010) J Stat Mech P07010 Dotsenko Klumov P03022 (2010).

2) WASEP

- T Sasamoto and H. Spohn PRL 104 230602 (2010) Nucl Phys B 834 523 (2010) J Stat Phys 140 209 (2010).
- G. Amir, I. Corwin, J. Quastel Comm.Pure.Appl.Math. 64 466 (2011)

- Continuum DP one free endpoint/KPZ Flat

- P. Calabrese, P. Le Doussal, ArXiv: 1104.1993 (2011).

Bethe ansatz details

$$-\psi'' - \bar{c}\delta(x)\psi(0) = E\psi$$

$$[\psi'/\psi]_{0^-}^{0^+} = -\bar{c}$$

n=2 $\quad H_2 = -\partial_{x_1}^2 - \partial_{x_2}^2 - \bar{c}\delta(x_1 - x_2)$

$$\psi(x) = \cos(kx) - \frac{\bar{c}}{2k}\text{sgn}(x)\sin(kx)$$

$$\psi(0) = 1$$

$$\psi_{\lambda_1,\lambda_2}(x_1, x_2) =$$

$$sym_{x_1,x_2}\; e^{i\lambda_1 x_1 + i\lambda_2 x_2} \left(1 - \frac{ic}{\lambda_2 - \lambda_1}\text{sgn}(x_2 - x_1)\right)$$

$$E = \lambda_1^2 + \lambda_2^2$$

Bethe ansatz details

$$-\psi'' - \bar{c}\delta(x)\psi(0) = E\psi$$

$$[\psi'/\psi]_{0^-}^{0^+} = -\bar{c}$$

n=2 $\quad H_2 = -\partial_{x_1}^2 - \partial_{x_2}^2 - \bar{c}\delta(x_1 - x_2)$

$$\psi(x) = \cos(kx) - \frac{\bar{c}}{2k}\mathrm{sgn}(x)\sin(kx)$$

$$\psi(0) = 1$$

$$\psi_{\lambda_1,\lambda_2}(x_1, x_2) = sym_{x_1,x_2}\, e^{i\lambda_1 x_1 + i\lambda_2 x_2}\left(1 - \frac{ic}{\lambda_2 - \lambda_1}\mathrm{sgn}(x_2 - x_1)\right)$$

$$E = \lambda_1^2 + \lambda_2^2$$

Periodic BC = Bethe equations $\quad e^{i\lambda_1 L} = \frac{\lambda_1 - \lambda_2 - i\bar{c}}{\lambda_1 - \lambda_2 + i\bar{c}}$

Bethe ansatz details

$$-\psi'' - \bar{c}\delta(x)\psi(0) = E\psi$$

$$[\psi'/\psi]_{0^-}^{0^+} = -\bar{c}$$

n=2 $\quad H_2 = -\partial_{x_1}^2 - \partial_{x_2}^2 - \bar{c}\delta(x_1 - x_2)$

$$\psi(x) = \cos(kx) - \frac{\bar{c}}{2k}\mathrm{sgn}(x)\sin(kx)$$

$$\psi(0) = 1$$

$$\psi_{\lambda_1,\lambda_2}(x_1,x_2) = sym_{x_1,x_2}\, e^{i\lambda_1 x_1 + i\lambda_2 x_2}\left(1 - \frac{ic}{\lambda_2 - \lambda_1}\mathrm{sgn}(x_2 - x_1)\right)$$

$$E = \lambda_1^2 + \lambda_2^2$$

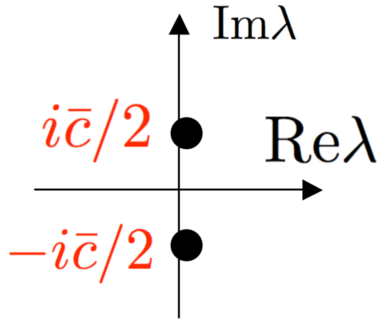
Periodic BC = Bethe equations $\quad e^{i\lambda_1 L} = \dfrac{\lambda_1 - \lambda_2 - i\bar{c}}{\lambda_1 - \lambda_2 + i\bar{c}}$

solutions:

- 2 1-string $\quad (\lambda_1, \lambda_2) = (k_1, k_2) \in R^2$

$$\lambda_j = \frac{2\pi n_j}{L} + o(\frac{1}{L})$$

- 1 2-string $\quad \lambda_{1,2} = k \pm i\dfrac{\bar{c}}{2} + O(ie^{-\bar{c}L})$



$$\overline{Z^n} = \langle x_0 \ldots x_0 | e^{-tH_{LL}} | x_0 \ldots x_0 \rangle$$

$$= \sum_\mu \frac{|\langle x_0 \ldots x_0 | \mu \rangle|^2}{||\mu||^2} e^{-tE_\mu}$$

all eigenstates are of the form

$$\Psi_\mu = \sum_P A_P \prod_{j=1}^n e^{i\lambda_{P_\ell} x_\ell}$$

$$A_P = \prod_{n \geq \ell > k \geq 1} \left(1 - \frac{ic \ \mathrm{sgn}(x_\ell - x_k))}{\lambda_{P_\ell} - \lambda_{P_k}}\right)$$

$$\overline{Z^n} = \langle x_0 \dots x_0 | e^{-tH_{LL}} | x_0 \dots x_0 \rangle$$

$$= \sum_{\mu} \frac{|\langle x_0 \dots x_0 | \mu \rangle|^2}{||\mu||^2} e^{-tE_\mu}$$

all eigenstates are of the form

$$\Psi_\mu = \sum_P A_P \prod_{j=1}^{n} e^{i \lambda_{P_\ell} x_\ell}$$

$$A_P = \prod_{n \geq \ell > k \geq 1} \left(1 - \frac{ic \ \mathrm{sgn}(x_\ell - x_k))}{\lambda_{P_\ell} - \lambda_{P_k}}\right)$$

Bethe equations + large L

All possible partitions of n into j=1,..ns strings each with mj particles

$$n = \sum_{j=1}^{n_s} m_j$$

$$\lambda^{j,a} = k_j + \frac{i\bar{c}}{2}(j + 1 - 2a) \qquad a = 1, \dots, m_j$$

$$E_\mu = \sum_{j=1}^{n_s} (m_j k_j^2 - \frac{\bar{c}^2}{12} m_j (m_j^2 - 1)) \qquad K_\mu = \sum_{j=1}^{n_s} m_j k_j$$

(Kardar) ground state ns=1, m1=n, k1=0

what is needed?

$$\overline{Z^n} = \sum_\mu \frac{|\langle x_0 \ldots x_0|\mu\rangle|^2}{||\mu||^2} e^{-tE_\mu}$$

$$\langle 0 \cdots 0|\mu\rangle = \Psi_\mu(0,..0) = n!$$

norm of states: Calabrese-Caux (2007)

$$||\mu||^2 = \frac{n!(L\bar{c})^{n_s}}{(\bar{c})^n} \frac{\prod_{j=1}^{n_s} m_j^2}{\Phi[k,m]}$$

$$\Phi[k,m] = \prod_{1\leq i<j\leq n_s} \frac{(k_i-k_j)^2 + (m_i-m_j)^2 c^2/4}{(k_i-k_j)^2 + (m_i+m_j)^2 c^2/4}$$

integer moments of partition sum

$$n = \sum_{j=1}^{n_s} m_j$$

$$\overline{\hat{Z}^n} = \sum_{n_s=1}^{n} \frac{n!}{n_s!(2\pi\bar{c})^{n_s}} \sum_{(m_1,\ldots m_{n_s})_n}$$

$$\int \prod_{j=1}^{n_s} \frac{dk_j}{m_j} \Phi[k,m] \prod_{j=1}^{n_s} e^{m_j^3 \frac{\bar{c}^2 t}{12} - m_j k_j^2 t},$$

$$\Phi[k,m] = \prod_{1\le i<j\le n_s} \frac{(k_i-k_j)^2 + (m_i-m_j)^2 c^2/4}{(k_i-k_j)^2 + (m_i+m_j)^2 c^2/4}$$

numerical check of second moment

$$Z_{\hat{x},\hat{t}+1} = (Z_{\hat{x}-\frac{1}{2},\hat{t}} + Z_{\hat{x}+\frac{1}{2},\hat{t}})e^{-\beta V_{\hat{x},\hat{t}+1}}$$

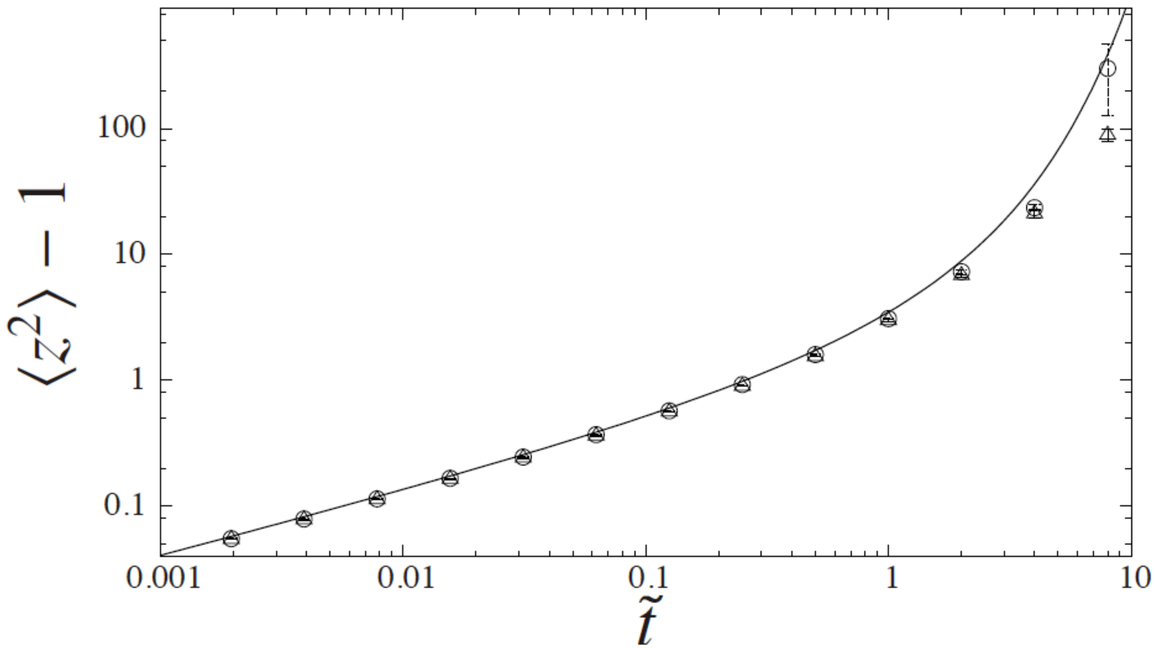
$$\kappa = 4T \qquad \tilde{x} = 4\hat{x}/T^2$$

$$\tilde{t} = 2\hat{t}/T^4$$

unit gaussian on the lattice $\bar{c} = 1$

$$\lambda = (\frac{\bar{c}^2}{4}t)^{1/3}$$

$$\overline{z^2} = 1 + \sqrt{2\pi}\lambda^{3/2}e^{2\lambda^3}(1 + \text{erf}(\sqrt{2}\lambda^{3/2}))$$



$$z = Z/\overline{Z}$$

FIG. 1: $\overline{z^2} - 1$ ($4\ 10^6$ samples) for $\hat{t} = 128$ (triangle), $\hat{t} = 256$ (circle) function of $\tilde{t}$ compared to formula (11) with $\bar{c} = 1$.

Numerical check, small time expansion

$$\overline{(\ln z)^2}^c = \sqrt{2\pi}\lambda^{3/2} + (4 + 5\pi - \frac{32\pi}{3\sqrt{3}})\lambda^3 + ..$$

$$\overline{(\ln z)^3}^c = (\frac{32}{3\sqrt{3}} - 6)\pi\lambda^3 + ..$$

$$\lambda = (\frac{\bar{c}^2}{4}t)^{1/3}$$

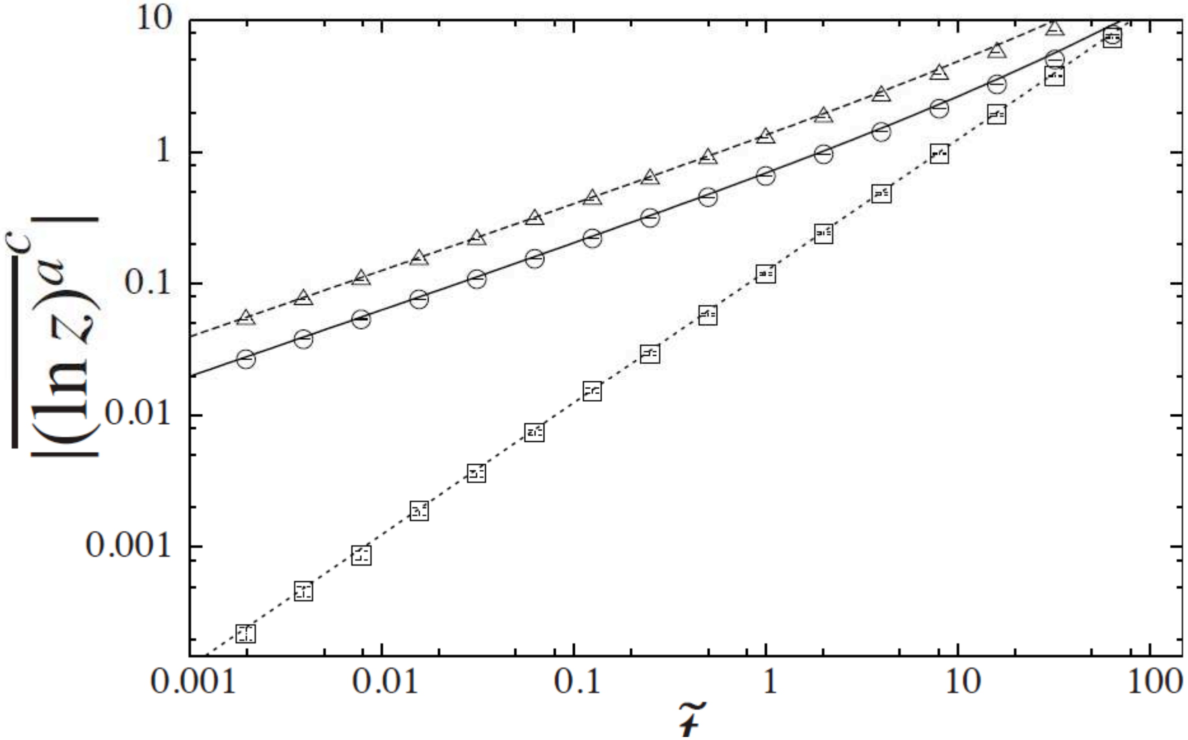


FIG. 2: From top to bottom the cumulants (4 10^6 samples) $\overline{(\ln z)^2}^c$ (dashed line, triangle), $-\overline{(\ln z)}$ (solid line, circle), and $\overline{(\ln z)^3}^c$ (dotted line, square) for $\hat{t} = 256$ as compared with the the analytical formula (12) with $\bar{c} = 1$.

generating function of moments

$$g(x) = 1 + \sum_{n=1}^{\infty} \frac{(-e^{\lambda x})^n}{n!} \overline{Z^n} = \overline{\exp(-e^{\lambda(x-f)})}$$

$$F = \lambda f$$

$$\lambda = (\frac{\bar{c}^2}{4} t)^{1/3}$$

$$\lim_{\lambda \to \infty} g(x) = \overline{\theta(f-x)} = Prob(f > x)$$

generating function of moments

$$F = \lambda f$$

$$\lambda = (\frac{\bar{c}^2}{4} t)^{1/3}$$

$$g(x) = 1 + \sum_{n=1}^{\infty} \frac{(-e^{\lambda x})^n}{n!} \overline{Z^n} = \overline{\exp(-e^{\lambda(x-f)})}$$

$$\lim_{\lambda \to \infty} g(x) = \overline{\theta(f-x)} = Prob(f > x)$$

reorganise sum over number of strings

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, x)$$

$$Z(n_s, x) = \sum_{m_1, \ldots m_{n_s}=1}^{\infty} \frac{(-1)^{\sum_j m_j}}{(4\pi\lambda^{3/2})^{n_s}}$$

$$\prod_{j=1}^{n_s} \int \frac{dk_j}{m_j} \prod_{1 \le i < j \le n_s} \frac{(k_i - k_j)^2 + (m_i - m_j)^2 \lambda^3}{(k_i - k_j)^2 + (m_i + m_j)^2 \lambda^3} \prod_{j=1}^{n_s} e^{\frac{1}{3}\lambda^3 m_j^3 - m_j k_j^2 + \lambda x m_j}$$

Interactions between strings

generating function of moments

$$g(x) = 1 + \sum_{n=1}^{\infty} \frac{(-e^{\lambda x})^n}{n!} \overline{Z^n} = \overline{\exp(-e^{\lambda(x-f)})}$$

$$F = \lambda f$$

$$\lambda = (\frac{\bar{c}^2}{4} t)^{1/3}$$

$$\lim_{\lambda \to \infty} g(x) = \overline{\theta(f-x)} = Prob(f > x)$$

reorganise sum over number of strings

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, x)$$

Airy trick

$$\int_{-\infty}^{\infty} dy Ai(y) e^{yw} = e^{w^3/3}$$

$$Z(n_s, x) = \sum_{m_1, \dots m_{n_s}=1}^{\infty} \frac{(-1)^{\sum_j m_j}}{(4\pi\lambda^{3/2})^{n_s}}$$

$$\prod_{j=1}^{n_s} \int \frac{dk_j}{m_j} \prod_{1 \le i < j \le n_s} \frac{(k_i - k_j)^2 + (m_i - m_j)^2 \lambda^3}{(k_i - k_j)^2 + (m_i + m_j)^2 \lambda^3} \prod_{j=1}^{n_s} e^{\frac{1}{3}\lambda^3 m_j^3 - m_j k_j^2 + \lambda x m_j}$$

Interactions between strings

One string contribution ns=1

$$Z(1,x) = \int_{v>0} \frac{dv\ v^{1/2}}{2\pi\lambda^{3/2}} dy Ai(y) \sum_{m=1}^{\infty} (-1)^m e^{\lambda m y - v m + \lambda x m}$$

$$v \to \lambda v$$

$$y \to y + v - x$$

$$Z(1,x) = -\int_{v>0} \frac{dv\ v^{1/2}}{2\pi} dy Ai(y+v-x) \frac{e^{\lambda y}}{1+e^{\lambda y}}$$

$$\frac{e^{\lambda y}}{1+e^{\lambda y}} \to \theta(y) \qquad \lim_{\lambda\to\infty} Z(1,x) = -\int_{w>0} \frac{dw}{3\pi} w^{3/2} Ai(w-x)$$

independent string approximation

$$g_{ind}(x) = \exp(Z(1,x)) \qquad Prob_{ind}(f > x) = g_{ind}(x)$$

correct tail for large negative f (exponent and prefactor..)

full solution

$$Z(n_s, x) = \sum_{m_1,\ldots m \quad =1}^{\infty} \frac{(-1)^{\sum_j m_j}}{(4\pi\lambda^{3/2})^{n_s}}$$

$$\prod_{j=1}^{n_s} \int \frac{dk_j}{m_j} \prod_{1\leq i<j\leq n_s} \frac{(k_i - k_j)^2 + (m_i - m_j)^2\lambda^3}{(k_i - k_j)^2 + (m_i + m_j)^2\lambda^3} \prod_{j=1}^{n_s} e^{\frac{1}{3}\lambda^3 m_j^3 - m_j k_j^2 + \lambda x m_j}$$

$$det[\frac{1}{i(k_i - k_j)\lambda^{-3/2} + (m_i + m_j)}]$$

$$= \prod_{i<j} \frac{(k_i - k_j)^2 + (m_i - m_j)^2\lambda^3}{(k_i - k_j)^2 + (m_i + m_j)^2\lambda^3} \prod_{i=1}^{n_s} \frac{1}{2m_i}$$

Result: Fredholm determinant

$$Z(n_s, x) = \int_{v_i>0} \prod_{i=1}^{n_s} dv_i \;\; det[K_x(v_i, v_j)]$$

$$\lambda = (\frac{\bar{c}^2}{4} t)^{1/3}$$

$$K_x(v, v') = \Phi_x(v + v', v - v')$$

$$\Phi_x(u, w) = - \int \frac{dk}{2\pi} dy Ai(y + k^2 - x + u) \frac{e^{\lambda y - ikw}}{1 + e^{\lambda y}}$$

Result: Fredholm determinant

$$Z(n_s, x) = \int_{v_i>0} \prod_{i=1}^{n_s} dv_i \;\; det[K_x(v_i, v_j)]$$

$$\lambda = (\frac{\bar{c}^2}{4} t)^{1/3}$$

$$K_x(v, v') = \Phi_x(v + v', v - v')$$

$$\Phi_x(u, w) = -\int \frac{dk}{2\pi} dy Ai(y + k^2 - x + u) \frac{e^{\lambda y - ikw}}{1 + e^{\lambda y}}$$

$$g(x) = Det[1 + P_0 K_x P_0]$$

P_s projector on $[s, +\infty[$

$$= e^{Tr \ln(1+K)} \quad = 1 + TrK + O(TrK^2)$$

$n_s = 1$ $\int_{v>0} K_x(v, v)$

$n_s = 2$

Large time limit and F2(s)

$$\lambda = \left(\frac{\bar{c}^2}{4} t\right)^{1/3}$$

$$\lambda = +\infty$$

$$Prob(f > x) = g(x) = \det(1 + P_{-\frac{x}{2}} \tilde{K} P_{-\frac{x}{2}})$$

$$\tilde{K}(v, v') = -\int_{y>0} \frac{dk}{2\pi} dy Ai(y + k^2 + v + v') e^{-ik(v-v')}$$

Airy function identity

$$\int dk Ai(k^2 + v + v') e^{ik(v-v')} = 2^{2/3} \pi Ai(2^{1/3} v) Ai(2^{1/3} v')$$

$$Prob(f > x = -2^{2/3} s) = Det(1 - P_s K_{Ai} P_s) = F_2(s)$$

$$K_{Ai}(v, v') = \int_{y>0} Ai(v + y) Ai(v' + y)$$

Strong universality at large time

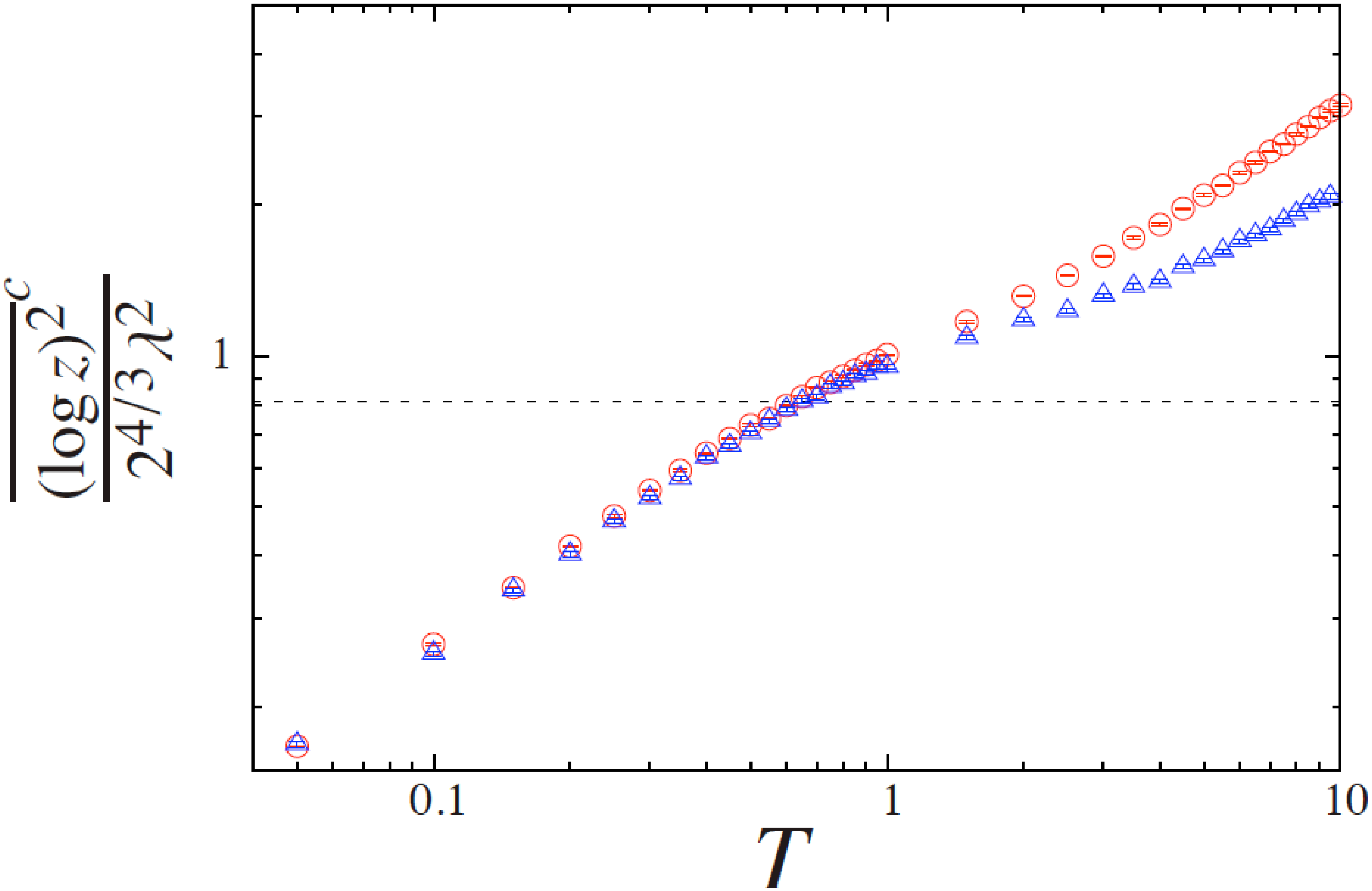


FIG. 3: $\overline{(\ln z)^2}^c/(2^{4/3}\lambda^2)$ plotted as a function of T, for increasing polymer length $\hat{t}$. Triangles correspond to $\hat{t} = 4096$, Circles to $\hat{t} = 256$ and the dotted line to the TW variance 0.81319... Averages are performed over 20000 samples.

conclusion

- continuum delta model describes DP high T
  strong universality

- solution using BA of DP fixed endpoints for all t
  (KPZ droplet init. cond).

- generating function is a Fredholm determinant for all t

- obtain free energy/KPZ height distribution for all t
  GUE confirmed large t = KPZ in KPZ class..

- solution using BA of DP one free endpoint for all t
  (KPZ flat init. cond).

An exact solution for the KPZ equation with flat initial conditions

P. Calabrese, P. Le Doussal,
ArXiv: 1104.1993 (2011), PRL to appear.

$$Z(n_s) = \sum_{m_i \geq 1} \prod_{j=1}^{n_s} \int_{k_j} \prod_{q=1}^{m_j} \frac{-2}{2ik_j + q} e^{\frac{\lambda^3}{3} m_j^3 - 4 m_j k_j^2 \lambda^3 - \lambda m_j s}$$

$$\times \mathrm{Pf}\left[\begin{pmatrix} \frac{2\pi}{2ik_i} \delta(k_i + k_j)(-1)^{m_i} \delta_{m_i, m_j} + \frac{1}{4}(2\pi)^2 \delta(k_i)\delta(k_j)(-1)^{\min(m_i, m_j)} \mathrm{sgn}(m_i - m_j) & \frac{1}{2}(2\pi)\delta(k_i) \\ -\frac{1}{2}(2\pi)\delta(k_j) & \frac{2ik_i + m_i - 2ik_j - m_j}{2ik_i + m_i + 2ik_j + m_j} \end{pmatrix} \right]$$

$$Z(n_s) = \prod_{j=1}^{n_s} \int_{v_j > 0} \mathrm{Pf}[\mathbf{K}(v_i, v_j)]_{2n_s, 2n_s}$$

$$g_\lambda(s) = \mathrm{Pf}[\mathbf{J} + \mathbf{K}] = \sum_{n_s=0}^{\infty} \frac{1}{n_s!} Z(n_s) \qquad \mathbf{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

$$K_{11} = \int_{y_1, y_2, k} Ai(y_1 + v_i + s + 4k^2) Ai(y_2 + v_j + s + 4k^2) [\frac{e^{-2i(v_i - v_j)k}}{2ik} f_{k/\lambda}(e^{\lambda(y_1+y_2)})$$

$$+ \frac{\pi\delta(k)}{2} F(2e^{\lambda y_1}, 2e^{\lambda y_2})]$$

$$K_{12} = \frac{1}{2} \int_y Ai(y + s + v_i)(e^{-2e^{\lambda y}} - 1)\ \delta(v_j)$$

$$K_{22} = 2\delta'(v_i - v_j)\,,$$

$$f_k(z) = \frac{-2\pi k z\, {}_1F_2\,(1; 2-2ik, 2+2ik; -z)}{\sinh(2\pi k)\,\Gamma\,(2-2ik)\,\Gamma\,(2+2ik)}\,, \qquad (19)$$

$$F(z_i, z_j) = \sinh(z_2 - z_1) + e^{-z_2} - e^{-z_1} + \int_0^1 du$$

$$\times J_0(2\sqrt{z_1 z_2 (1-u)})[z_1 \sinh(z_1 u) - z_2 \sinh(z_2 u)]\,.$$

$$g_\lambda(s) = \mathrm{Pf}[\mathbf{J}+\mathbf{K}] = \sum_{n_s=0}^{\infty} \frac{1}{n_s!} Z(n_s) \qquad \mathbf{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

$$K_{11} = \int_{y_1,y_2,k} Ai(y_1+v_i+s+4k^2)Ai(y_2+v_j+s+4k^2)[\frac{e^{-2i(v_i-v_j)k}}{2ik} f_{k/\lambda}(e^{\lambda(y_1+y_2)})$$

$$K_{12} = \frac{1}{2}\int_y Ai(y+s+v_i)(e^{-2e^{\lambda y}}-1)\ \delta(v_j) \qquad + \frac{\pi\delta(k)}{2} F(2e^{\lambda y_1}, 2e^{\lambda y_2})]$$

$$K_{22} = 2\delta'(v_i - v_j)\,,$$

$$f_k(z) = \frac{-2\pi k z\, {}_1F_2\,(1;2-2ik,2+2ik;-z)}{\sinh(2\pi k)\,\Gamma(2-2ik)\,\Gamma(2+2ik)}\,, \qquad (19)$$

$$F(z_i,z_j) = \sinh(z_2-z_1) + e^{-z_2} - e^{-z_1} + \int_0^1 du$$

$$\times J_0(2\sqrt{z_1 z_2(1-u)})[z_1\sinh(z_1 u) - z_2\sinh(z_2 u)]\,.$$

$$\lim_{\lambda\to+\infty} f_{k/\lambda}(e^{\lambda y}) = -\theta(y)$$

$$\lim_{\lambda\to+\infty} F(2e^{\lambda y_1}, 2e^{\lambda y_2}) =$$
$$\theta(y_1+y_2)(\theta(y_1)\theta(-y_2) - \theta(y_2)\theta(-y_1))$$

$$\lim_{\lambda\to+\infty} Z(n_s) = (-1)^{n_s}\int_{x_1,..x_{n_s}} \det[\mathcal{B}_s(x_i,x_j)]_{n_s\times n_s}\,.$$

$$g_\infty(s) = F_1(s) = \det[I - \mathcal{B}_s] \qquad \text{GOE Tracy Widom}$$

$$\mathcal{B}_s = \theta(x)Ai(x+y+s)\breve{\theta}(y)$$