Diffusing Predators Hunting a Diffusing Prey

16th Itzykson Meeting, Saclay, Jun 14-17, 2011

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Basic question: What is the survival probability of a diffusing prey that is hunted by diffusing predators?

Fact: one dimension most interesting $S_N(t) \sim t^{-\beta_N}$

Outline:

- 1 & 2 predators *exactly soluble*
- 3 predators *accurate exponent by electrostatics*
- N ≫ 1 predators $\beta_N \simeq \frac{1}{4} \ln N$
- N =∞ predators $S(t) \simeq \exp[-\frac{1}{8} \ln^2(t)]$
- simple argument for iterated logarithm law
- survival in wedges, cones, paraboloids

Dimension Dependence

d>2: hunt *unsuccessful*; the prey may survive forever

Polya (1921), Bramson & Griffeath (1991)

d=2: hunt *successful*, but hunters still essentially independent

$$\rightarrow S_N(t) \sim [S_1(t)]^N$$

d=1: *successful* hunt $S_N(t) \sim t^{-\beta_N}$

- *surrounded prey:* adding hunters is efficient → $\beta_N \sim N$
- *chased prey:* adding hunters is inefficient → slow decay

β_N sublinear in N

effective correlation between hunters

One Diffusing Hunter, One Stationary Prey

probability distribution of the prey-hunter separation:

$$p(x,t) = \frac{1}{\sqrt{4\pi Dt}}\left[e^{-(x-x_0)^2/4Dt} - e^{-(x+x_0)^2/4Dt}\right]$$

x_0 ≡ initial separation of hunter and prey
x ≡ current separation of hunter and prey

survival probability; integrate over all x:

$$S_1(t) = \text{erf}\left(\frac{x_0}{\sqrt{4Dt}}\right) \sim \frac{x_0}{\sqrt{\pi Dt}} \quad \text{as } t \to \infty$$

$$S_1 \sim t^{-\beta_1} \quad \text{with} \quad \beta_1 = \frac{1}{2}$$

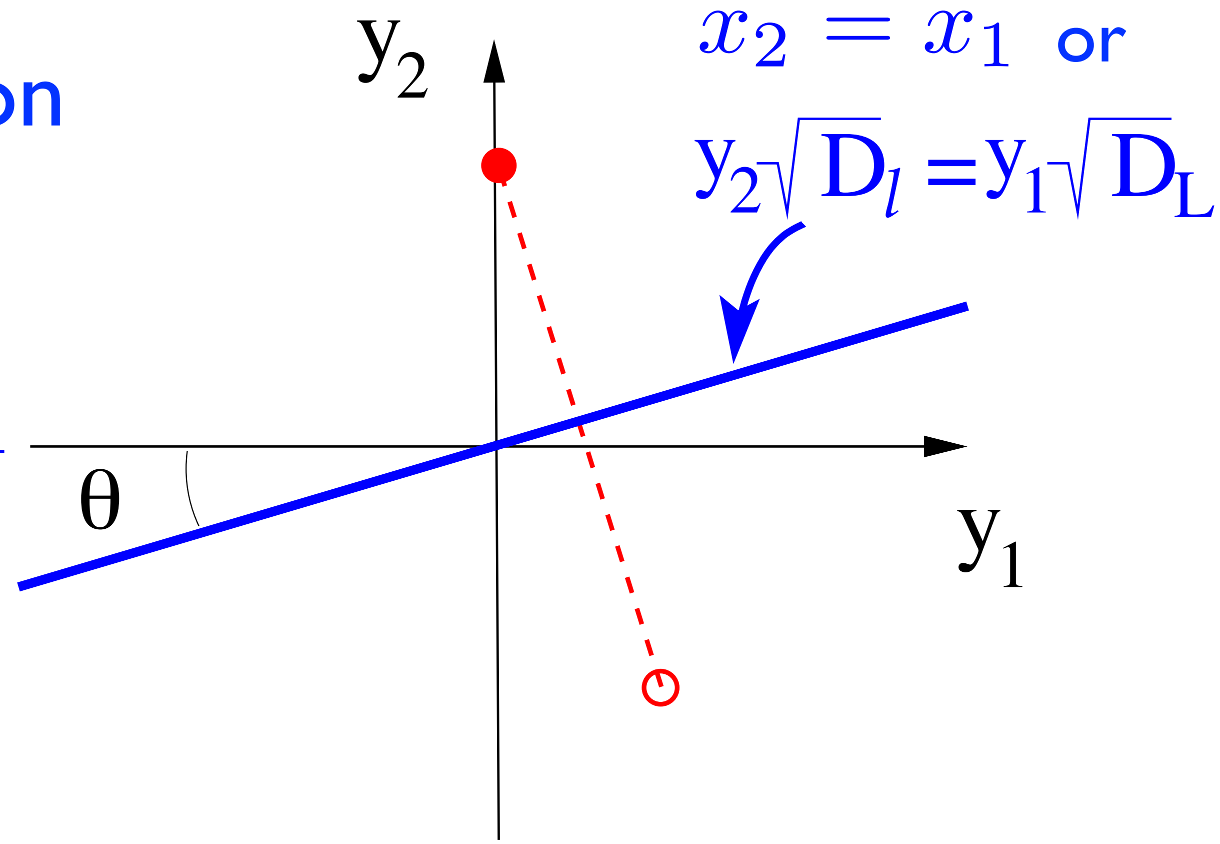
One Diffusing Hunter, One Diffusing Prey

x_1 < x_2

map to *isotropic* 2d diffusion

$$y_1 = x_1/\sqrt{D_L}$$
$$y_2 = x_2/\sqrt{D_\ell}$$

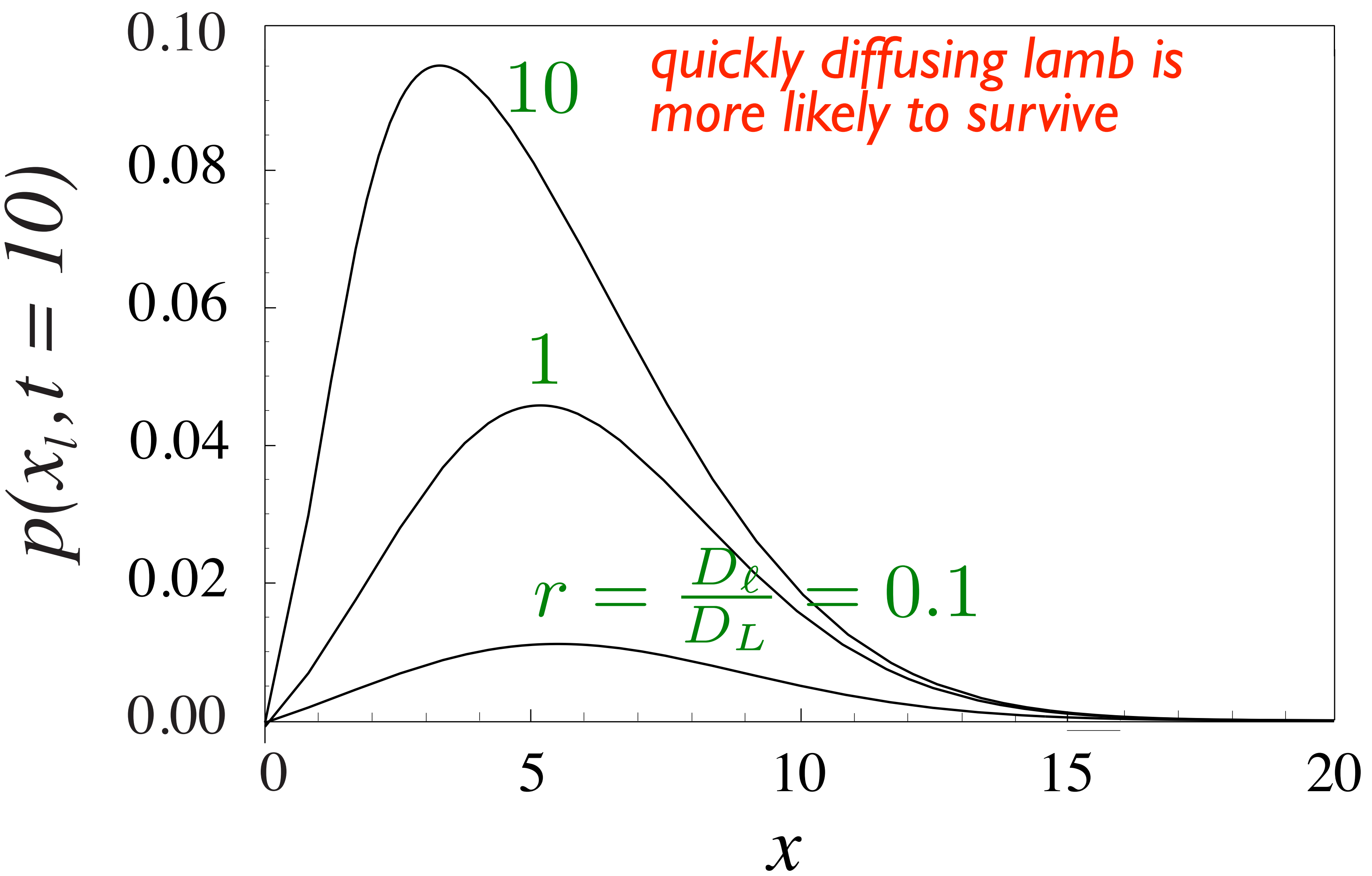
$$\theta = \tan^{-1}\sqrt{\frac{D_L}{D_\ell}}$$



$$p(y_1, y_2, t) = \frac{1}{4\pi t}\Big[e^{-[y_1^2+(y_2-\sqrt{D_\ell})^2]/4t} - e^{-[(y_1-\sqrt{D_\ell}\sin 2\theta)^2+(y_2+\sqrt{D_\ell}\cos 2\theta)^2]/4t}\Big]$$

Prey Probability Distribution

$$p(x_\ell, t) = \frac{1}{\sqrt{16\pi D_\ell t}} \left[e^{-(x_\ell - 1)^2/4D_\ell t} \operatorname{erfc}\left(-\frac{x_\ell \cot\theta}{\sqrt{4D_\ell t}}\right) - e^{-(x_\ell + \cos 2\theta)^2/4D_\ell t} \operatorname{erfc}\left(\frac{\sin 2\theta - x_\ell \cot\theta}{\sqrt{4D_\ell t}}\right) \right]$$



Two Diffusing Hunters, One Diffusing Prey

x_1

x_2

x_3

require $x_2 < x_3$ *and* $x_1 < x_3$

map to 3d diffusion

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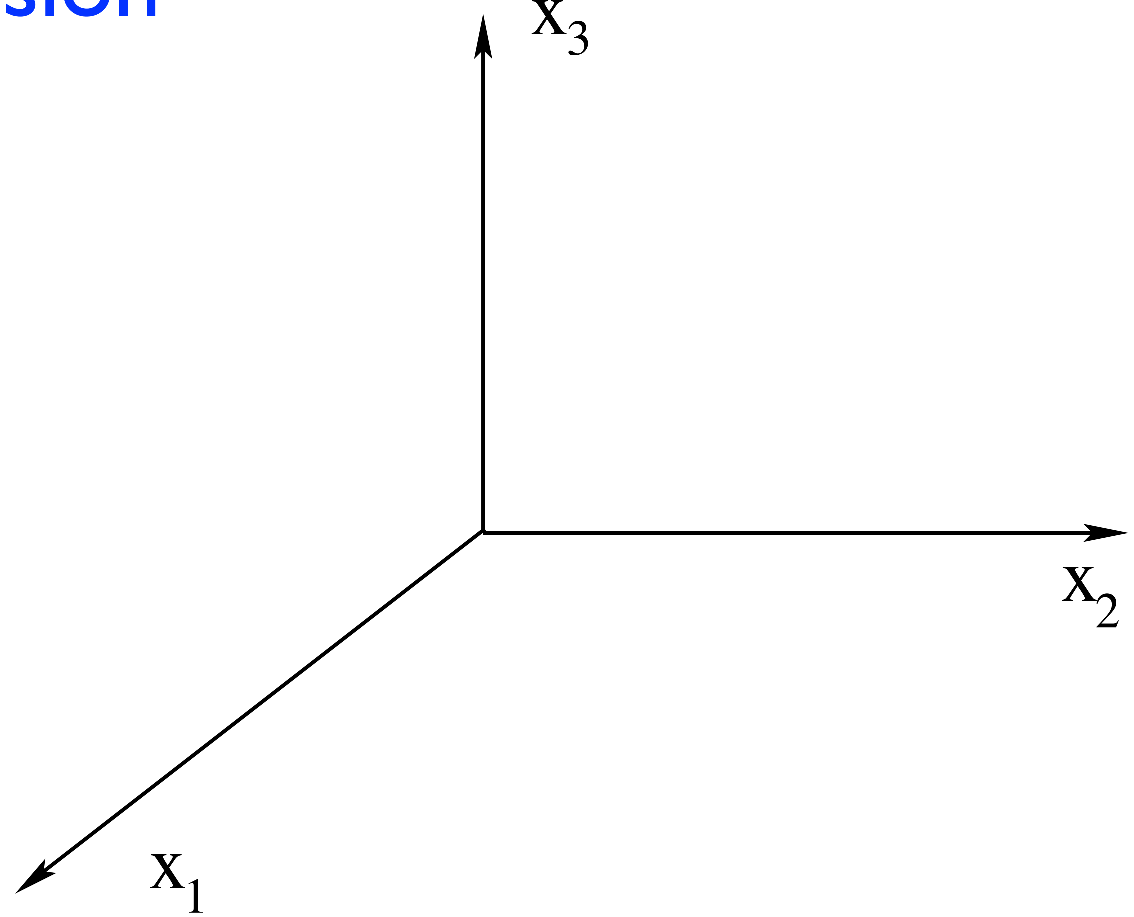

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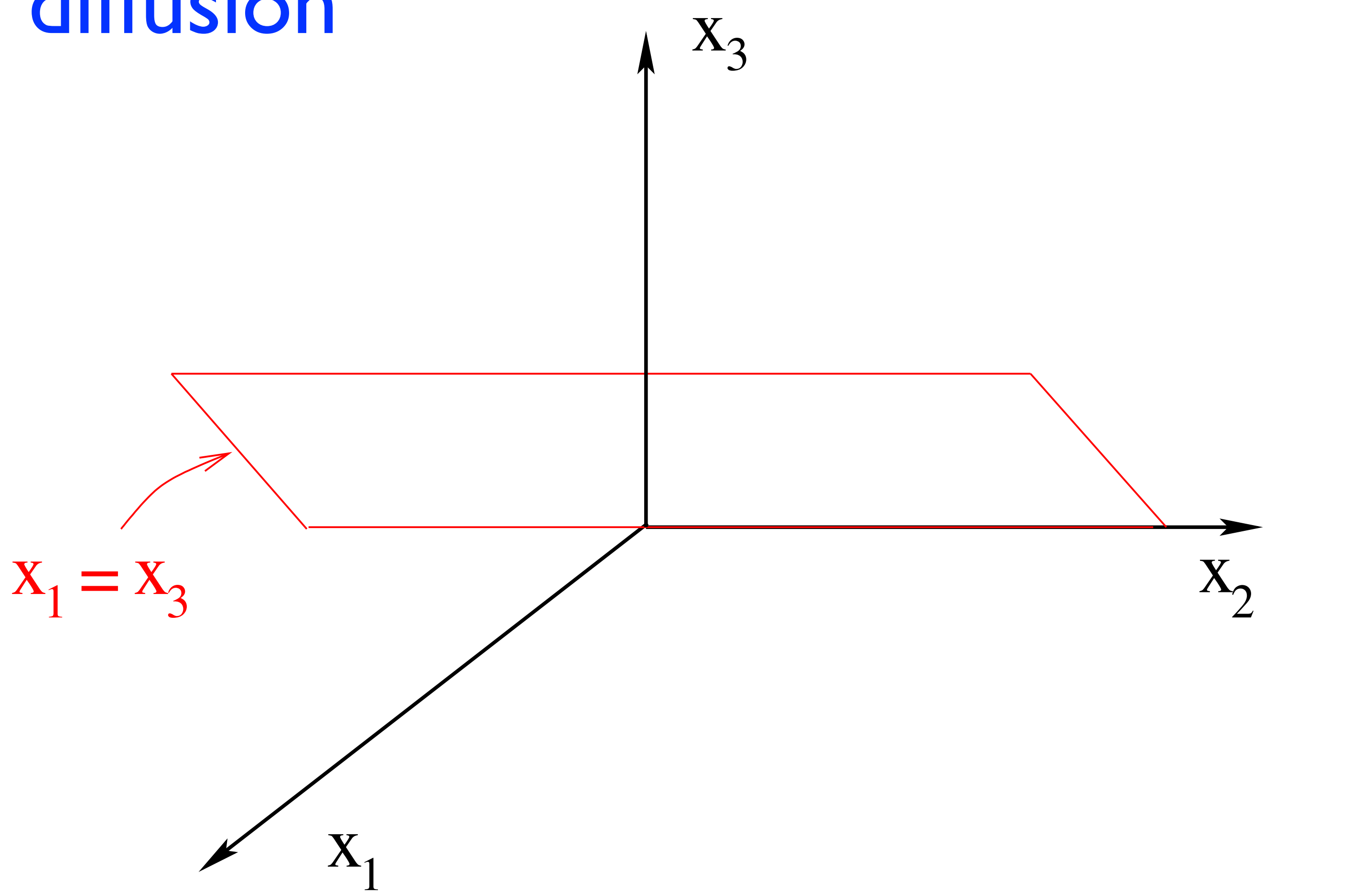

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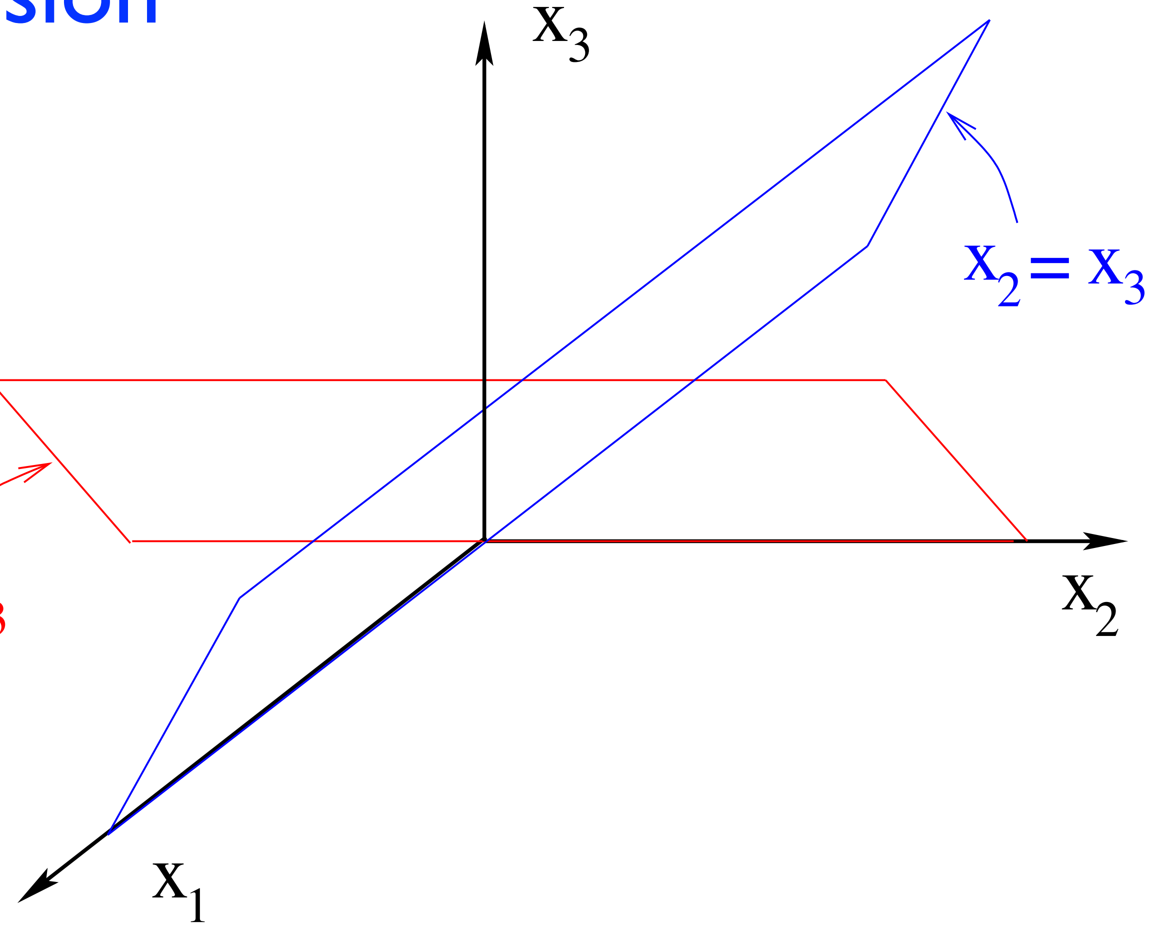

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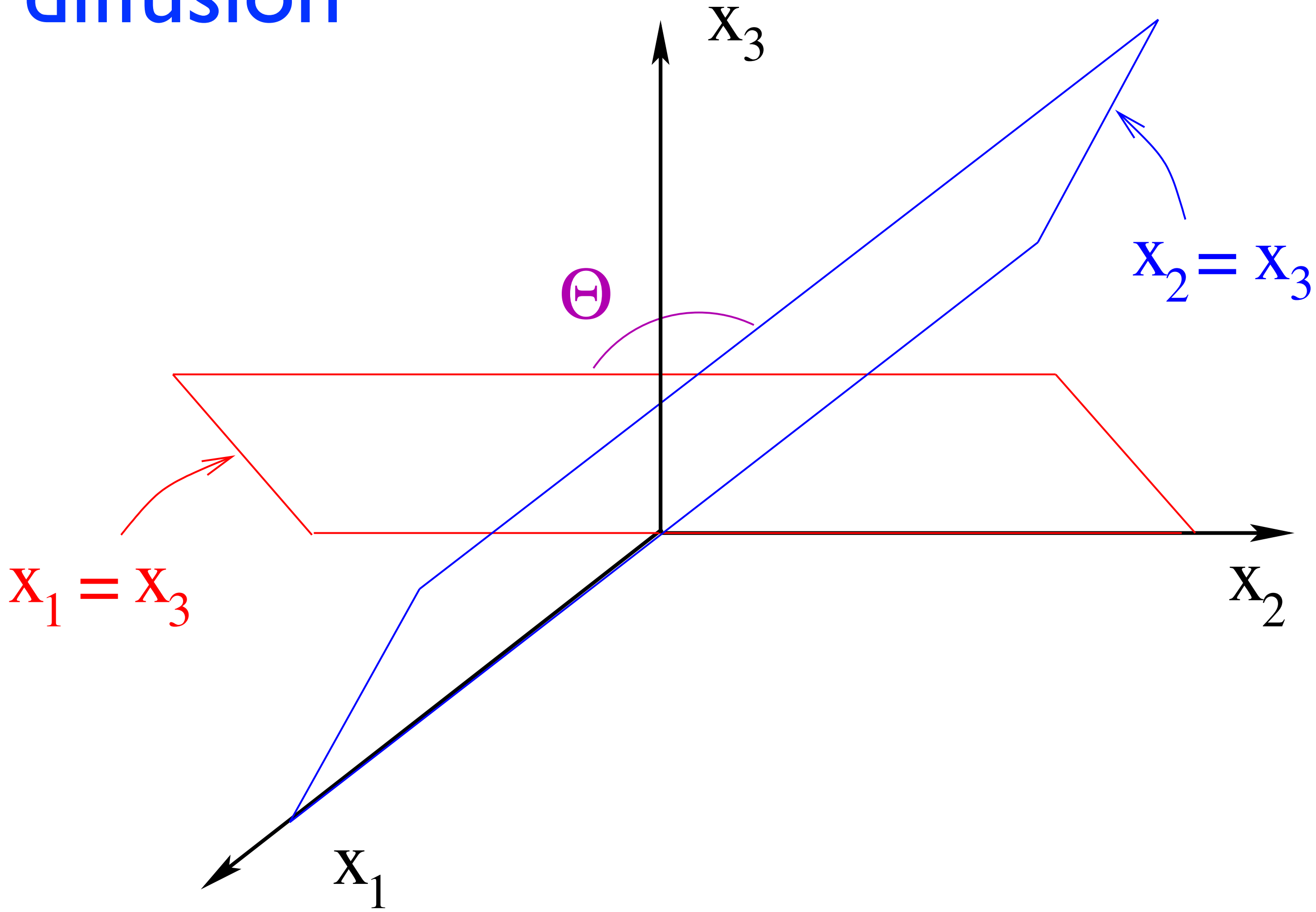
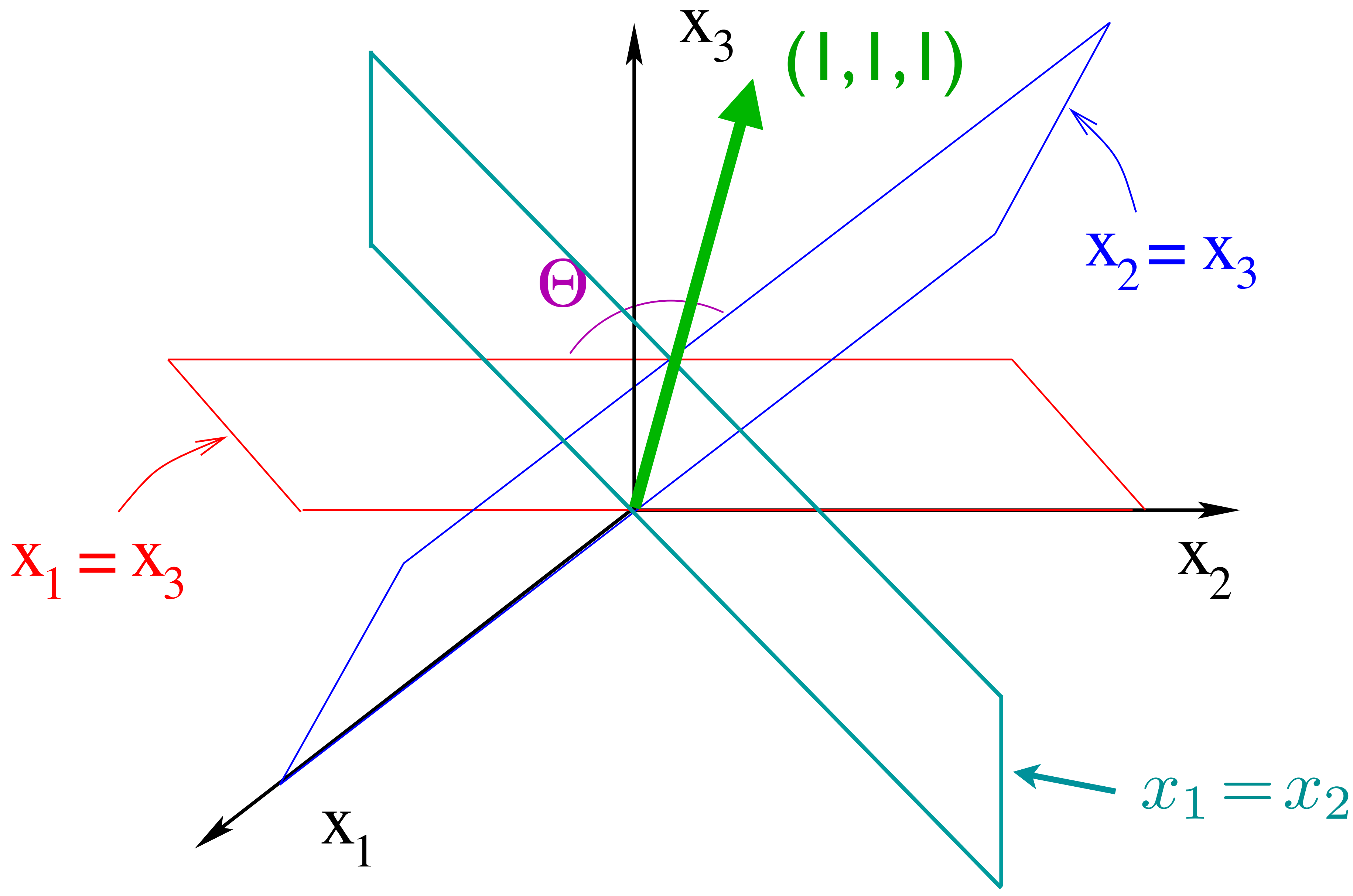
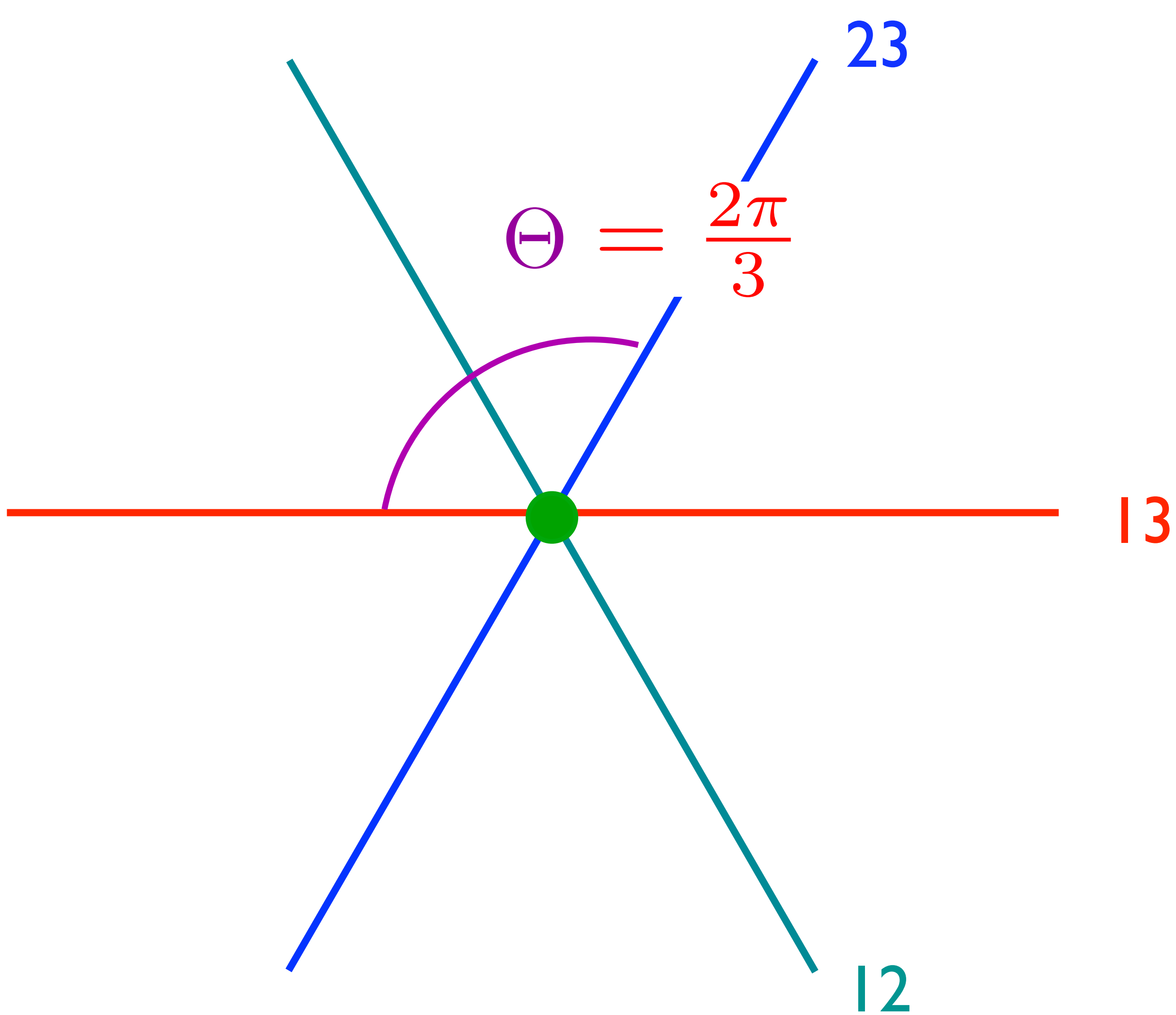


Equivalence to Diffusion in Absorbing Wedge

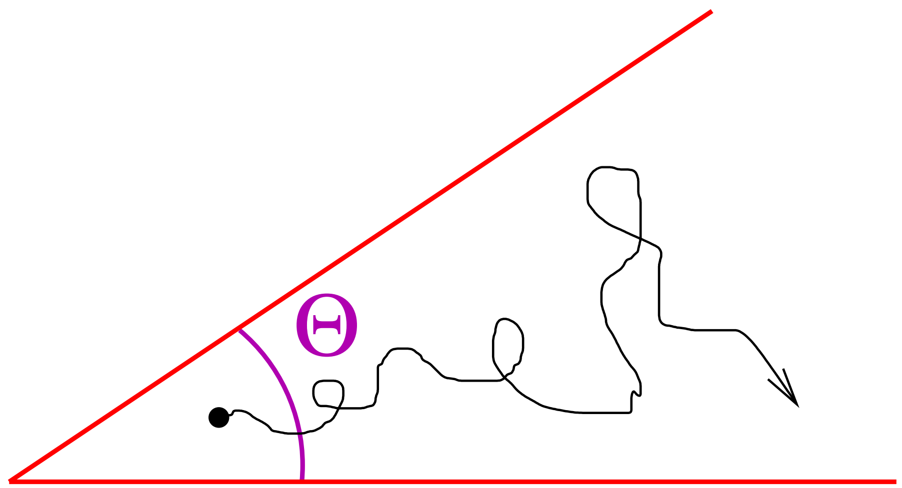


Equivalence to Diffusion in Absorbing Wedge

view perpendicular to (1,1,1)



Diffusion in Absorbing Wedge



$$S(t) \sim t^{-\pi/2\Theta}$$

Carslaw & Jaeger (1959)

electrostatic equivalence

$$\phi(r) \sim r^{-\pi/\Theta}$$

equivalent to injecting diffusing particles at fixed rate

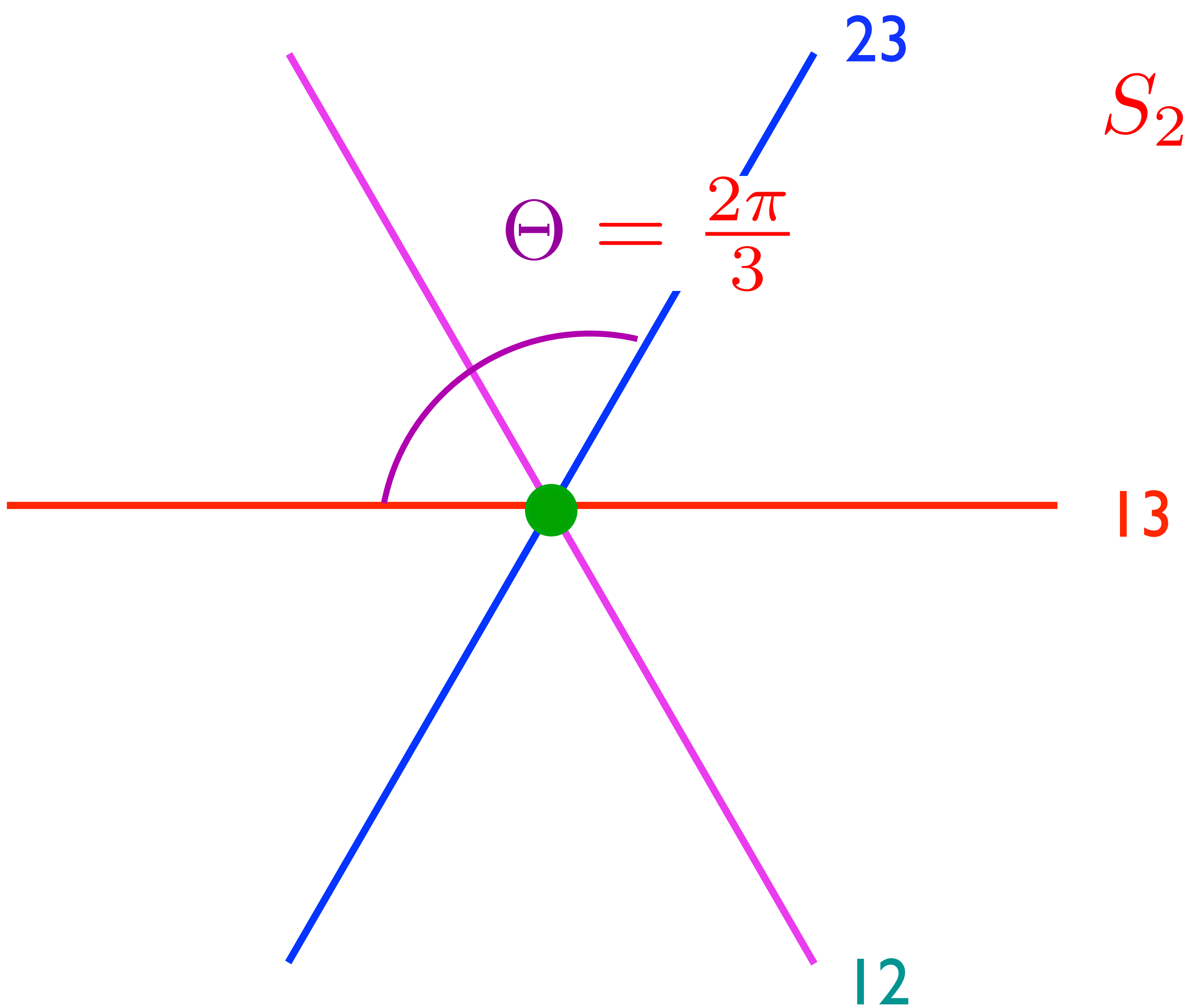
number of particles within wedge

$$N(t) \sim \int_0^{\Theta} d\theta \int_0^{\sqrt{Dt}} \phi(r)\, r\, dr$$

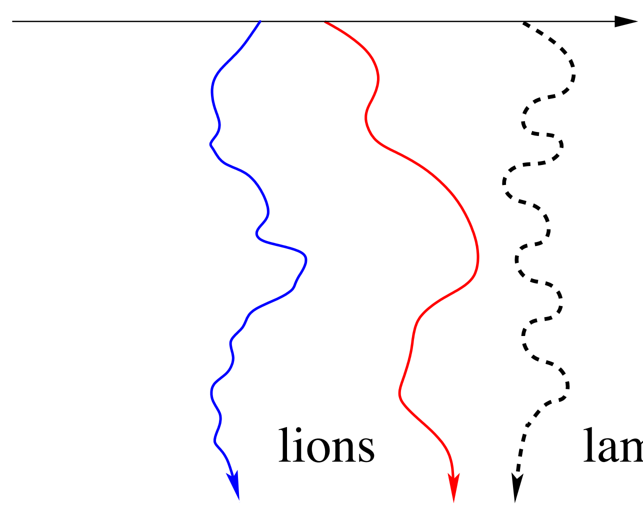
$$\sim \int_0^{\sqrt{Dt}} r^{1-\pi/\Theta} dr$$

$$\sim t^{1-\pi/2\Theta}$$

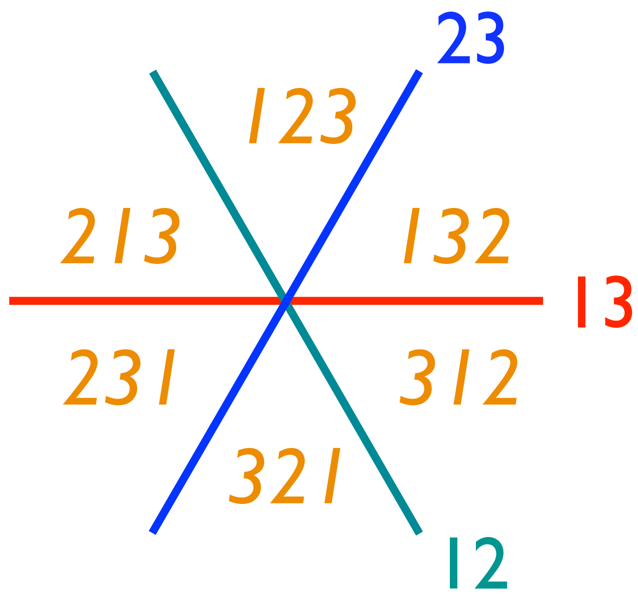
Equivalence to Diffusion in Absorbing Wedge



$S_2(t) \sim t^{-3/4}$ *chased prey*



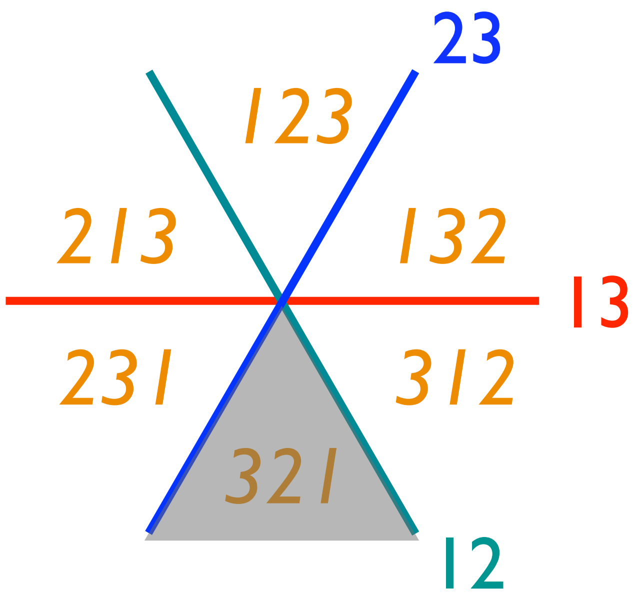
Systematics of the Equivalence



Systematics of the Equivalence

no reversal, 123 $\not\Rightarrow$ *321*

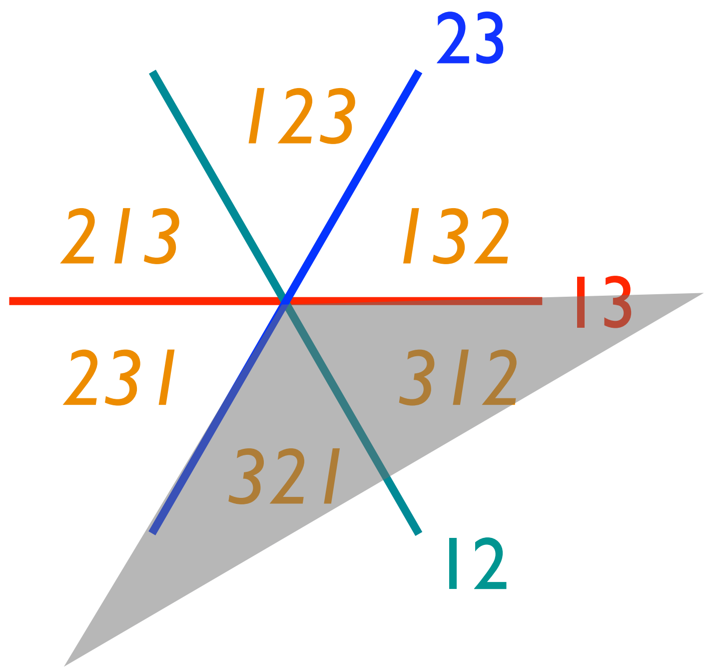
$t^{-3/10}$



Systematics of the Equivalence

no reversal, $123 \not\Rightarrow 321$ $\quad t^{-3/10}$

3 never trails, $123 \not\Rightarrow 321, 312$ $\quad t^{-3/8}$

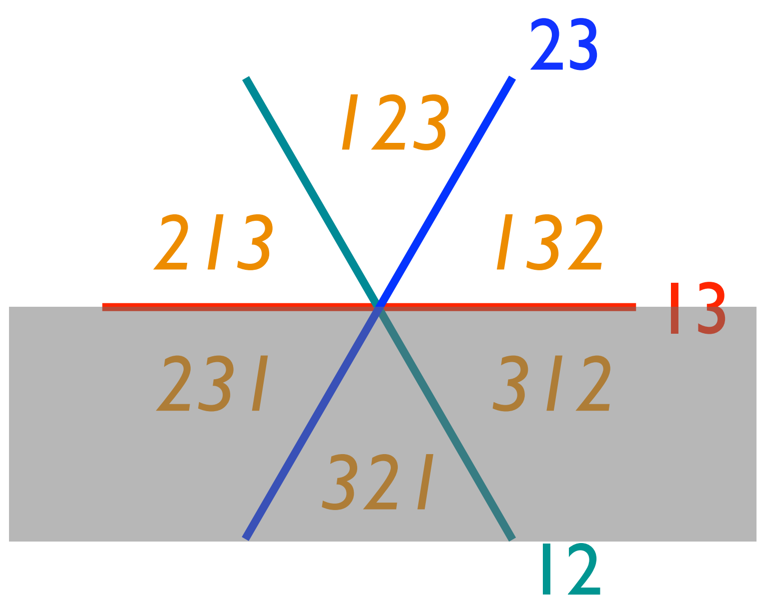


Systematics of the Equivalence

no reversal, $123 \not\Rightarrow 321$ $t^{-3/10}$

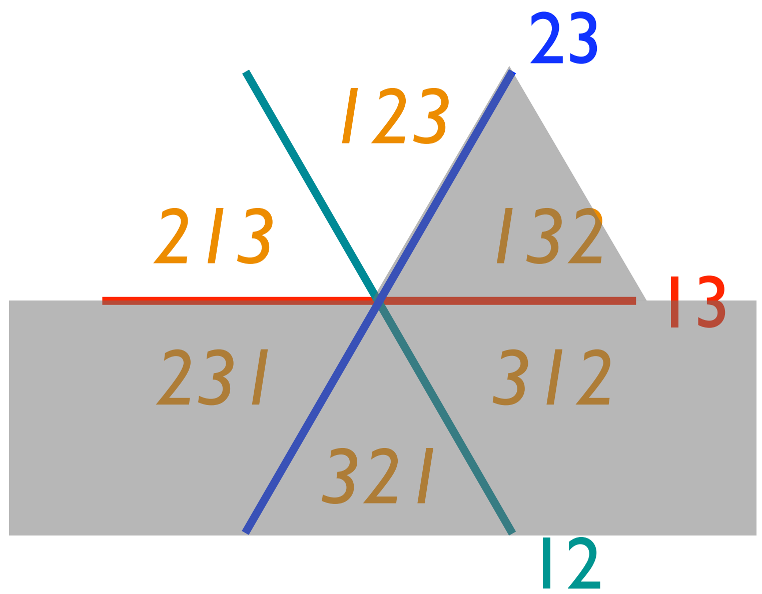
3 never trails, $123 \not\Rightarrow 321, 312$ $t^{-3/8}$

3 always leads 1 $t^{-3/6}$



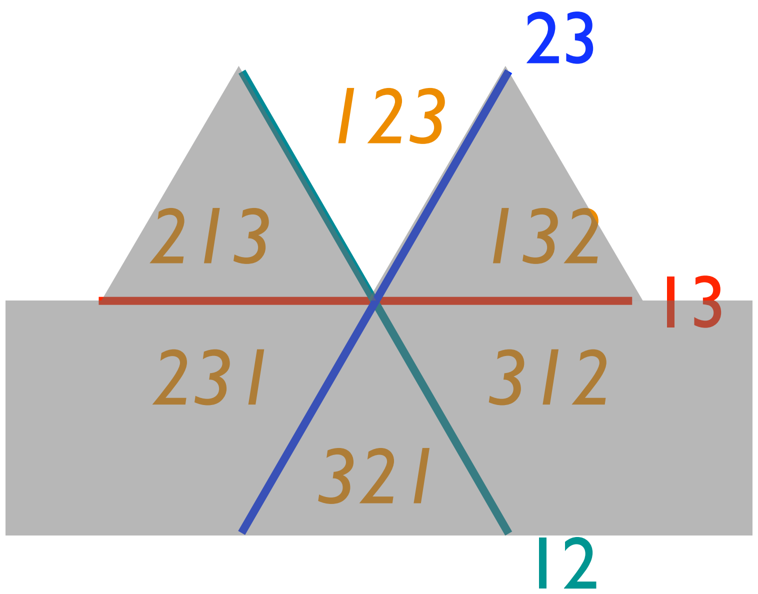
Systematics of the Equivalence

no reversal, 123 $\not\Rightarrow$ 321	$t^{-3/10}$
3 never trails, 123 $\not\Rightarrow$ 321, 312	$t^{-3/8}$
3 always leads 1	$t^{-3/6}$
3 always leads 1 & 2	$t^{-3/4}$



Systematics of the Equivalence

no reversal, $123 \not\Rightarrow 321$	$t^{-3/10}$
3 never trails, $123 \not\Rightarrow 321, 312$	$t^{-3/8}$
3 always leads 1	$t^{-3/6}$
3 always leads 1 & 2	$t^{-3/4}$
order preserved	$t^{-3/2}$



Three Diffusing Hunters, One Diffusing Prey

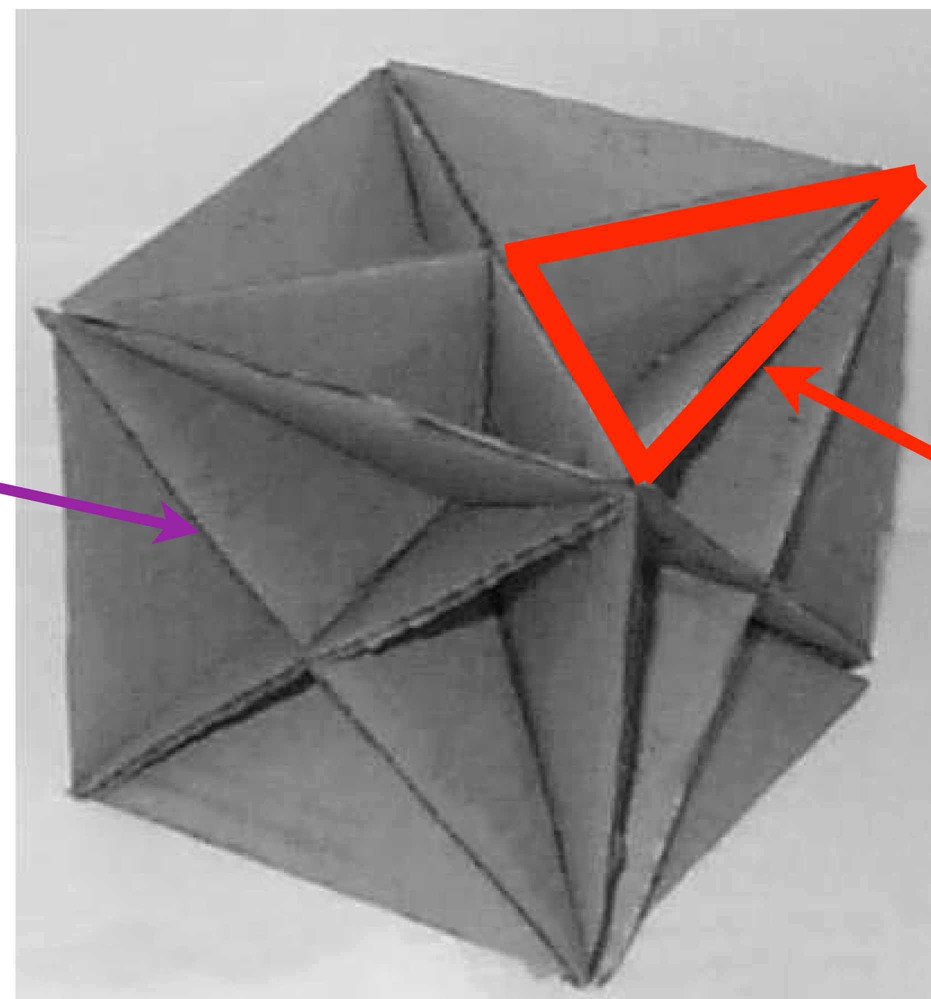
x_1 x_2 x_3 x_4

require $x_1 < x_4$, $x_2 < x_4$, $x_3 < x_4$

projection of 4-space onto 3d hyperplane ⊥ to (1,1,1,1)

construction by
D. ben-Avraham

each plane represents
$x_i = x_j$



each ***Weyl chamber***
represents one specific
ordering; for vicious
random walks
$x_1<x_2<x_3<x_4$

Three Diffusing Hunters, One Diffusing Prey

x_1

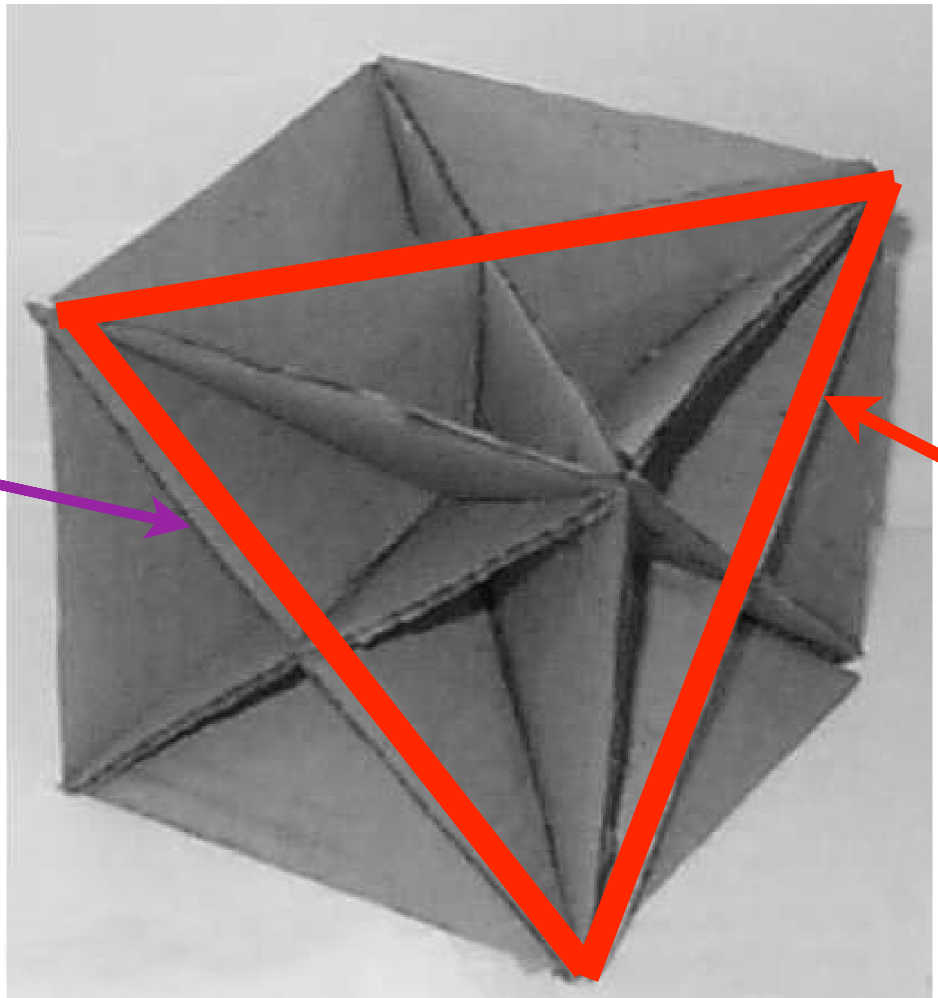
x_2

x_3

x_4

require $x_1 < x_4$, $x_2 < x_4$, $x_3 < x_4$

projection of 4-space onto 3d hyperplane $\perp$ to (1,1,1,1)



construction by D. ben-Avraham

each plane represents $x_i = x_j$

Weyl chamber for lamb survival: $x_1, x_2, x_3 < x_4$

Three Diffusing Hunters, One Diffusing Prey

x_1

x_2

x_3

x_4

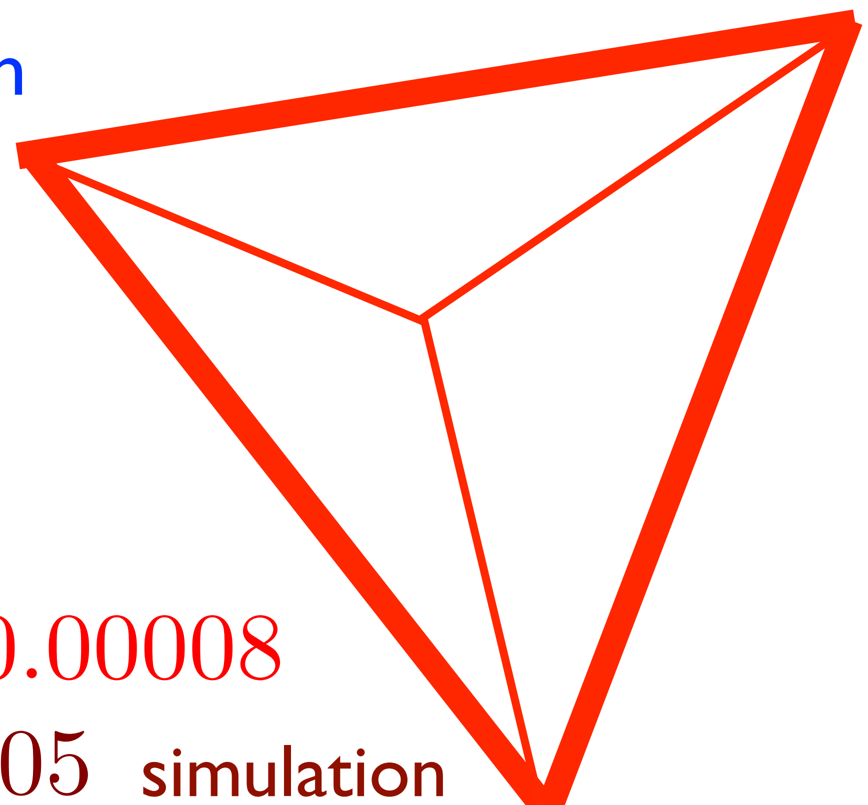
require $x_1 < x_4, \; x_2 < x_4, \; x_3 < x_4$

potential in Weyl chamber: $\phi(r) \sim r^{-\mu}$ $\quad \mu = 2.82684 \pm 0.00016$

(ben-Avraham et al. 2003)

electrostatics ↔ diffusion

$$\beta = \tfrac{1}{2}(\mu - N + 2)$$



$$\rightarrow \beta_3 = 0.91342 \pm 0.00008$$

0.913 ± 0.005 simulation

Many Diffusing Hunters, One Diffusing Prey

Krapivsky & SR (96, 99)

Many Diffusing Hunters, One Diffusing Prey

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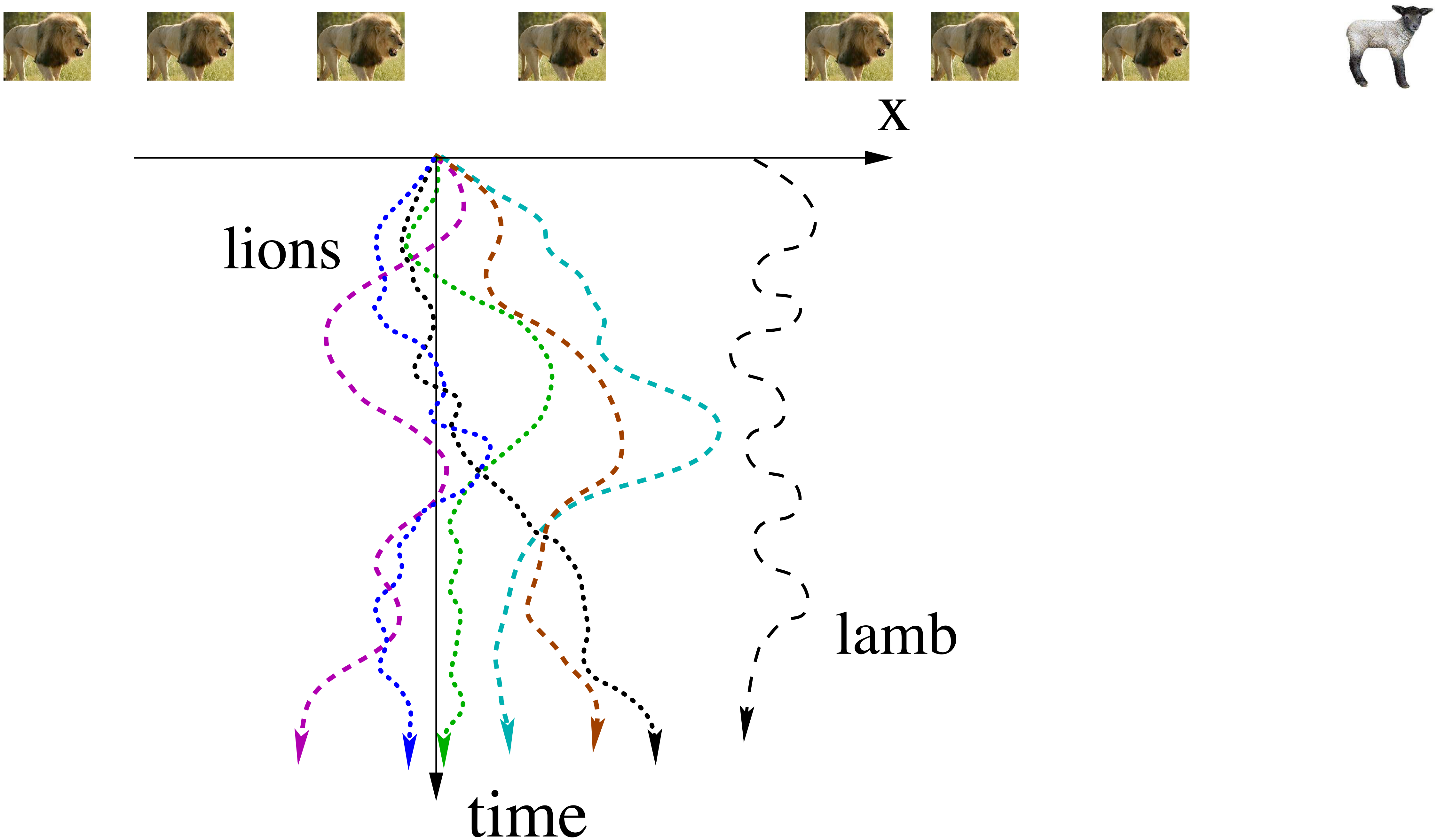

Many Diffusing Hunters, One Diffusing Prey

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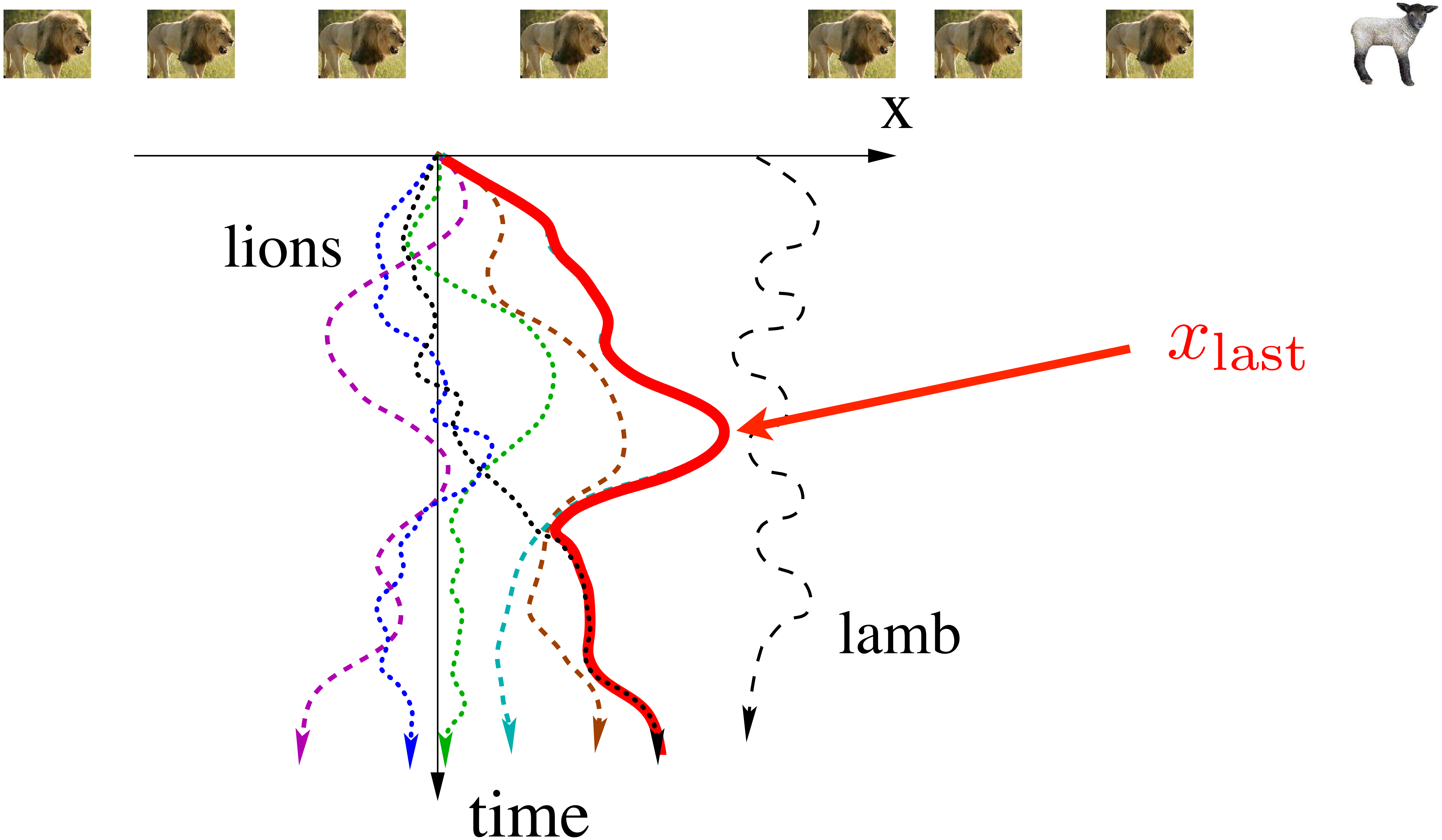

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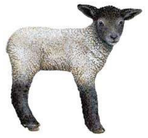
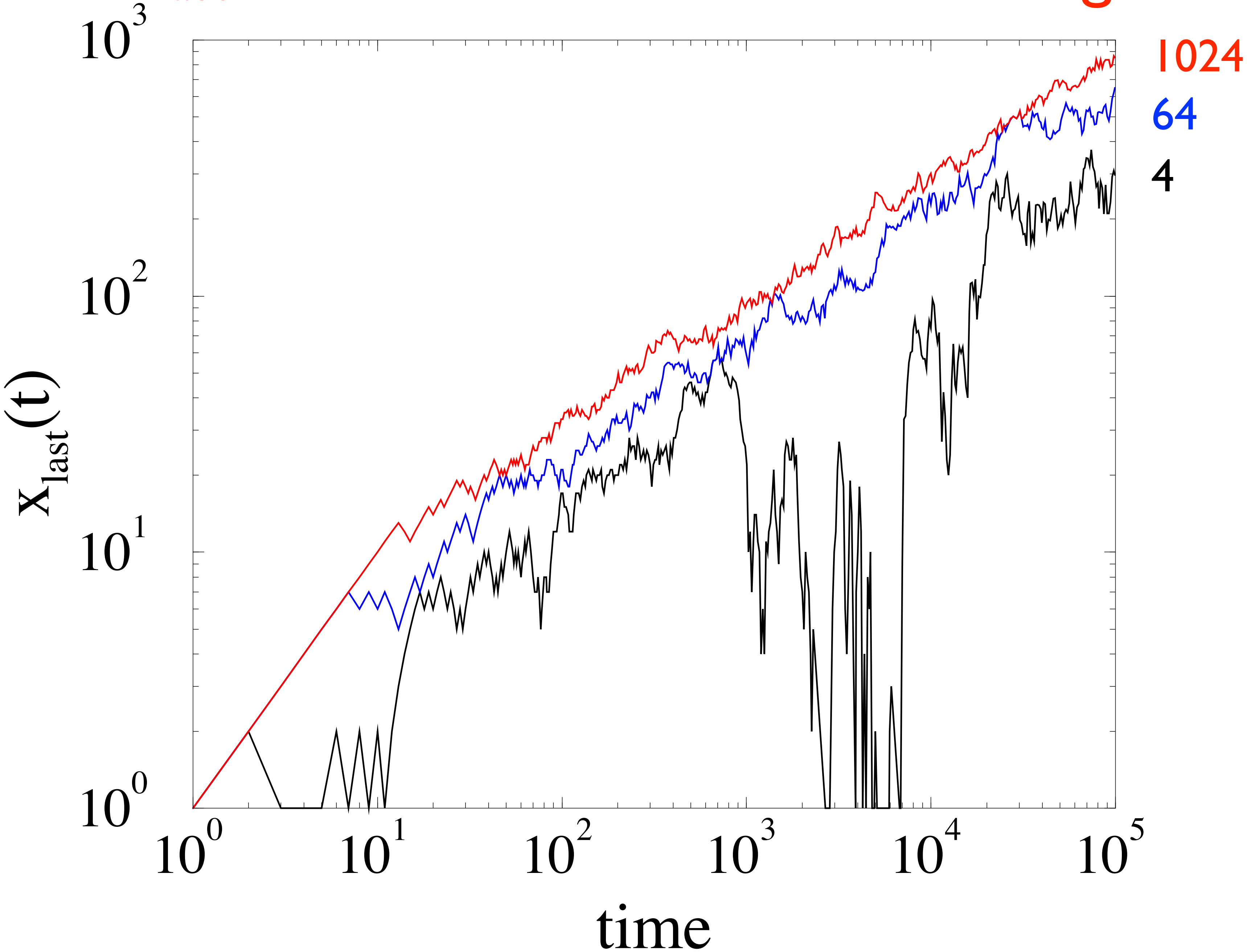


x_{last} becomes deterministic for large N



Many Diffusing Hunters, One Diffusing Prey

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extremal criterion for $x_{\rm last}$:

$$\int_{x_{\rm last}}^{\infty} \frac{N}{\sqrt{4\pi Dt}}\, e^{-x^2/4Dt}\, dx = 1$$

$$x_{\rm last}(t) \sim \sqrt{(4D\ln N)\, t} \equiv \sqrt{A_N\, t}$$

N≫1 lions

$$x_{\rm last}(t) \sim \sqrt{2D\ln(c_0^2 Dt)\, t}$$

N=∞ lions

constant density for x<0

only $N \sim c_0\sqrt{Dt}$ are dangerous

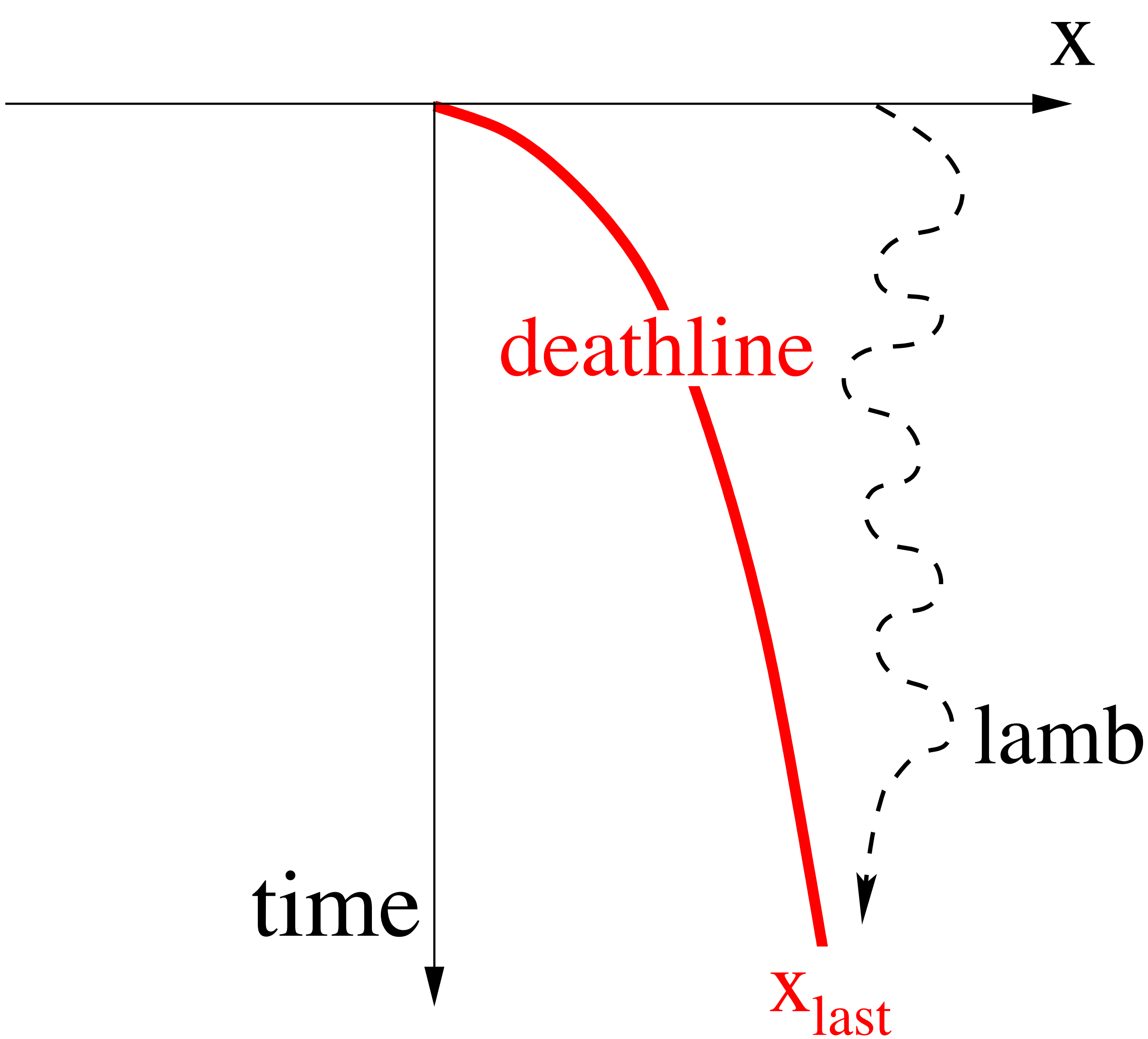
Many Diffusing Hunters, One Diffusing Prey

Krapivsky & SR (96, 99)

Effective Problem: Deterministic Deathline

$$x_{\rm last}(t) \sim \sqrt{(4D \ln N)\, t} \equiv \sqrt{A_N\, t}$$

$$x_{\rm last}(t) \sim \sqrt{2D \ln(c_0^2 D t)\, t}$$



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lamb probability distribution:

$$\frac{\partial p(x,t)}{\partial t} - \frac{x_{\text{last}}}{2t}\frac{\partial p(x,t)}{\partial x} = D\,\frac{\partial^2 p(x,t)}{\partial x^2} \qquad (0 \le x < \infty)$$

scaling ansatz: $p(x,t) \sim t^{-\beta_N - 1/2}\,F(\xi) \qquad \xi = x/x_{\text{last}}$

$$\frac{D}{A_N}\frac{d^2F}{d\xi^2} + \frac{\xi}{2}\frac{dF}{d\xi} + \left(\beta_N + \tfrac{1}{2}\right)F = 0$$

convert to Schrödinger equation:

define: $F(\xi) = e^{-\eta^2/4}\,\mathcal{D}(\eta) \quad \eta = \xi\sqrt{A_N/2D}$

$$\longrightarrow \quad \frac{d^2\mathcal{D}_{2\beta_N}}{d\eta^2} + \left[2\beta_N + \tfrac{1}{2} - \tfrac{1}{4}\eta^2\right]\mathcal{D}_{2\beta_N} = 0$$

subject to $\mathcal{D}_{2\beta_N}(\eta) = 0 \quad \begin{matrix} \eta = \sqrt{A_N/2D} \\ \eta = \infty \end{matrix}$

lowest energy eigenvalue determined by the criterion: $2\beta_N + \tfrac{1}{2} \simeq \eta^2/4$

$$\longrightarrow \beta_N \simeq A_N/16D$$

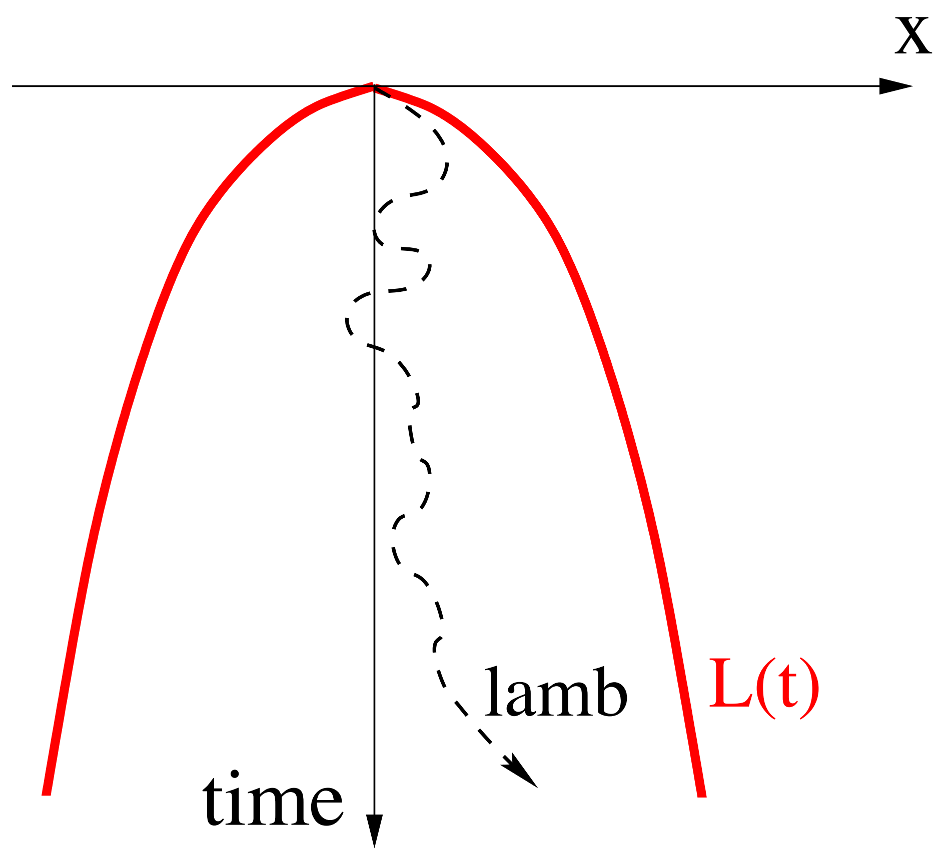
Summary

$$\beta_N \simeq \begin{cases} \frac{1}{4}\ln N & N \text{ finite} \\ \\ \frac{1}{8}\ln t & N = \infty \end{cases} \qquad S_\infty(t) \sim \exp\left(-\tfrac{1}{8}\,\ln^2 t\right)$$

Khintchine Iterated Logarithm Law

Khintchine (1924), Feller (1968), SR (01)

what is L(t) so that the lamb "survives"?



if $L(t) > t^{1/2}$, $\quad S_\infty > 0$ lamb survives

"free" approximation: $c(x,t) = \frac{S(t)}{\sqrt{4\pi Dt}}\, e^{-x^2/4Dt}$

$$\frac{\partial S}{\partial t} = -2D\frac{\partial c}{\partial x}\Big|_{x=L} = -\sqrt{\frac{L^2}{4\pi Dt}}\, e^{-L^2/4Dt}\,\frac{S}{t}$$

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if $L = \sqrt{At}$ (and $A \gg 1$)

$$\ln S(t) = -\int_0^t \beta\,\frac{dt'}{t'} \qquad \beta = \sqrt{\frac{A}{4\pi D}}\, e^{-A/4D}$$

if A=const., even if A ≫ 1 → lamb dies

if $L = \sqrt{4Dt\,f(t)}$

$$\ln S(t) = -\int_0^t \sqrt{\frac{f(t')}{\pi}}\, e^{-f(t')}\,\frac{dt'}{t'}$$

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if $L = \sqrt{4Dt\, f(t)}$

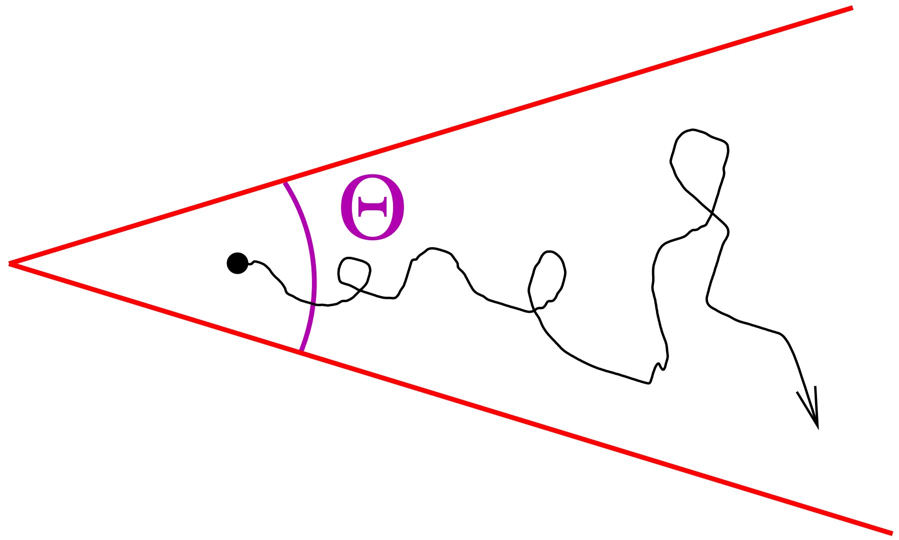
$$\ln S(t) = -\int_0^t \sqrt{\frac{f(t')}{\pi}} e^{-f(t')}\frac{dt'}{t'} = -\int_0^{\ln t} \sqrt{\frac{f(x)}{\pi}} e^{-f(x)}\, dx$$

for $f(x) = \lambda \ln x, \quad \rightarrow \quad \ln S \sim -\int^{\ln t}\frac{dx}{x^\lambda}$ $\quad$ marginal case $\lambda = 1$

when $L = \sqrt{4Dt\,\ln\ln t} \quad \rightarrow \quad S(t) \sim \frac{1}{\ln t}$

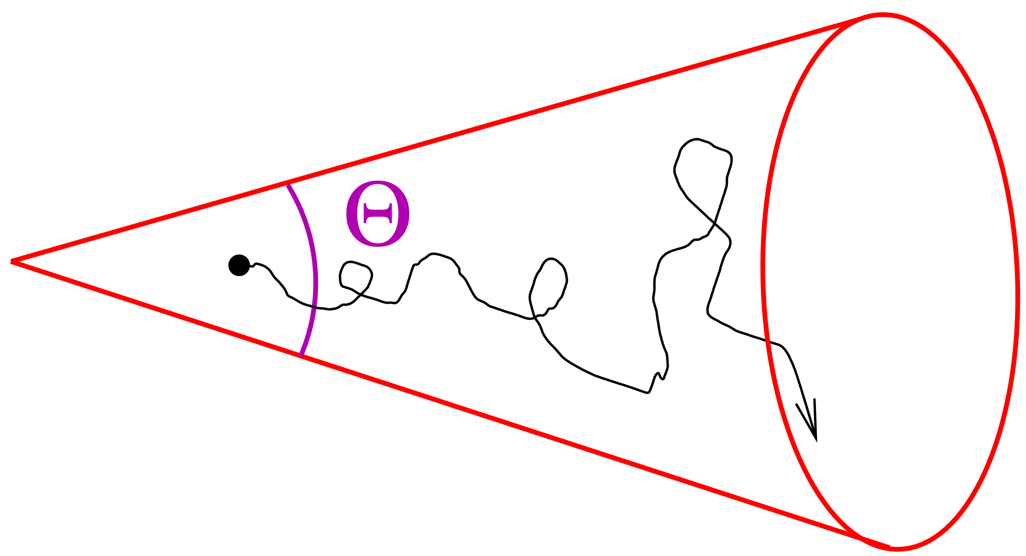
ultimately $L(t) \sim \sqrt{4Dt(\ln\ln t + \frac{3}{2}\ln\ln\ln t + \ldots)} \quad \rightarrow S(t) \sim \frac{1}{\ln\ln\ln\ldots\ln t}$

Survival in Wedges



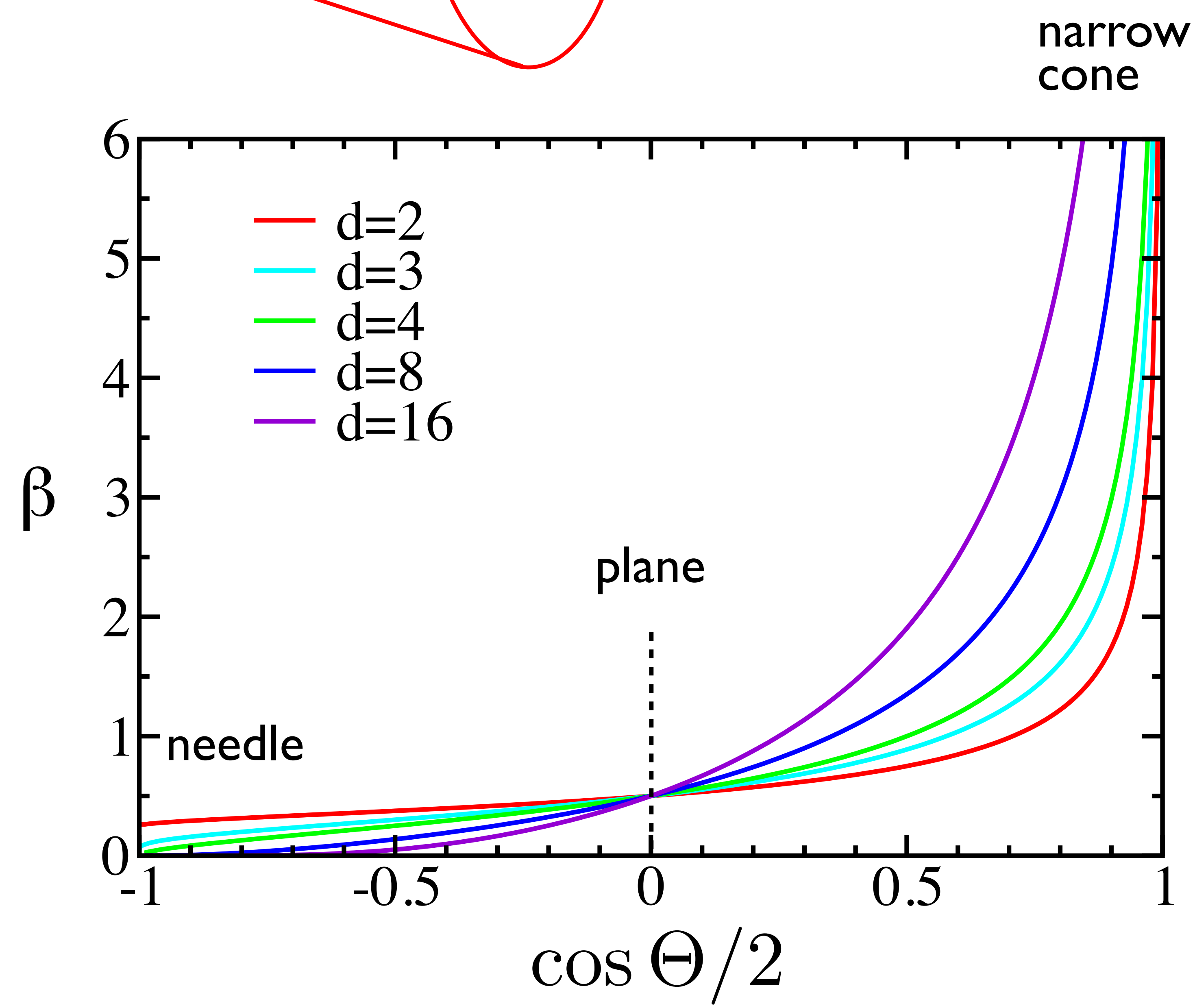
$$S(t) \sim t^{-\pi/2\Theta}$$

Survival in Wedges, Cones

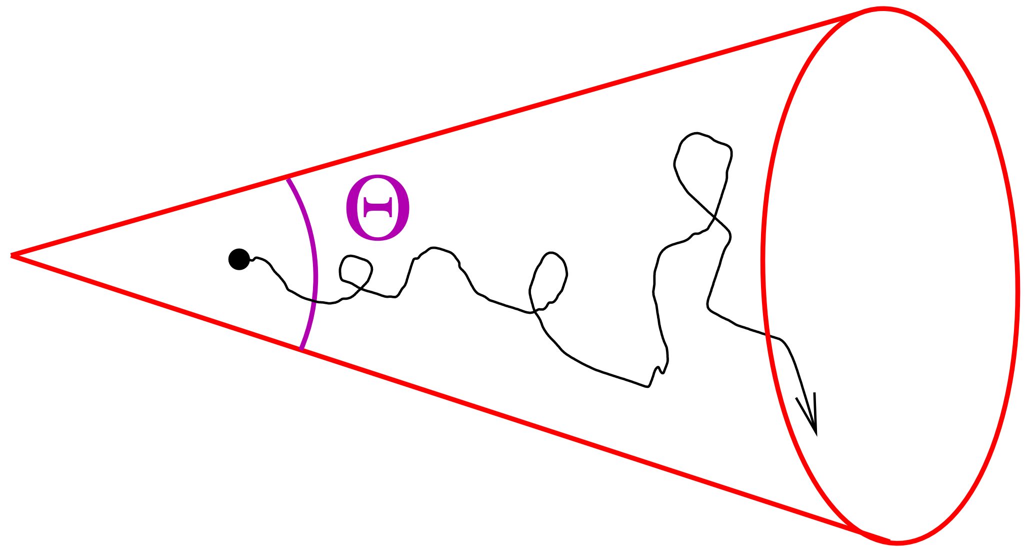


$$S(t) \sim t^{-\beta(\Theta,d)}$$

Ben-Naim and Krapivsky (2010)



Survival in Wedges, Cones



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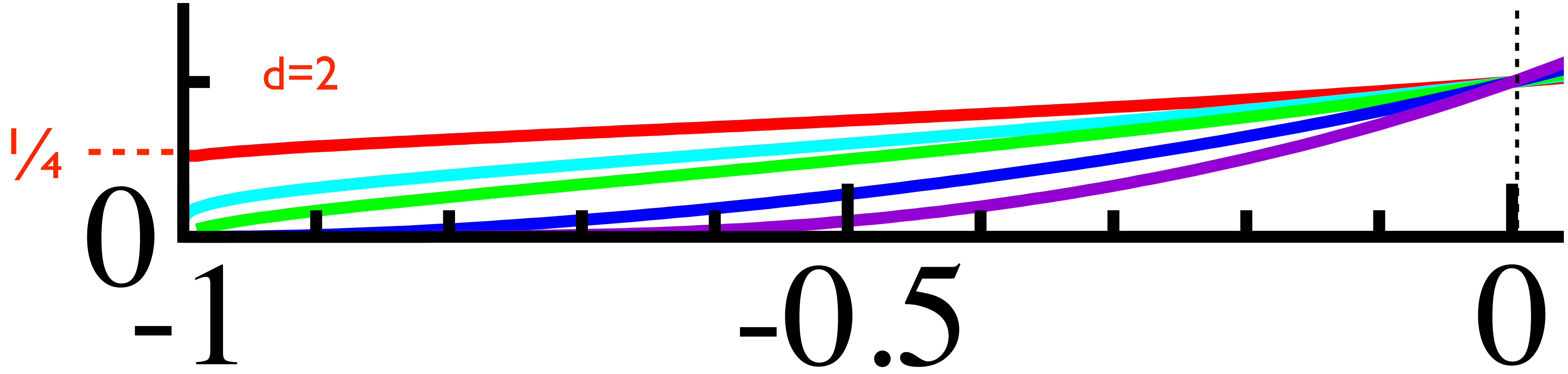
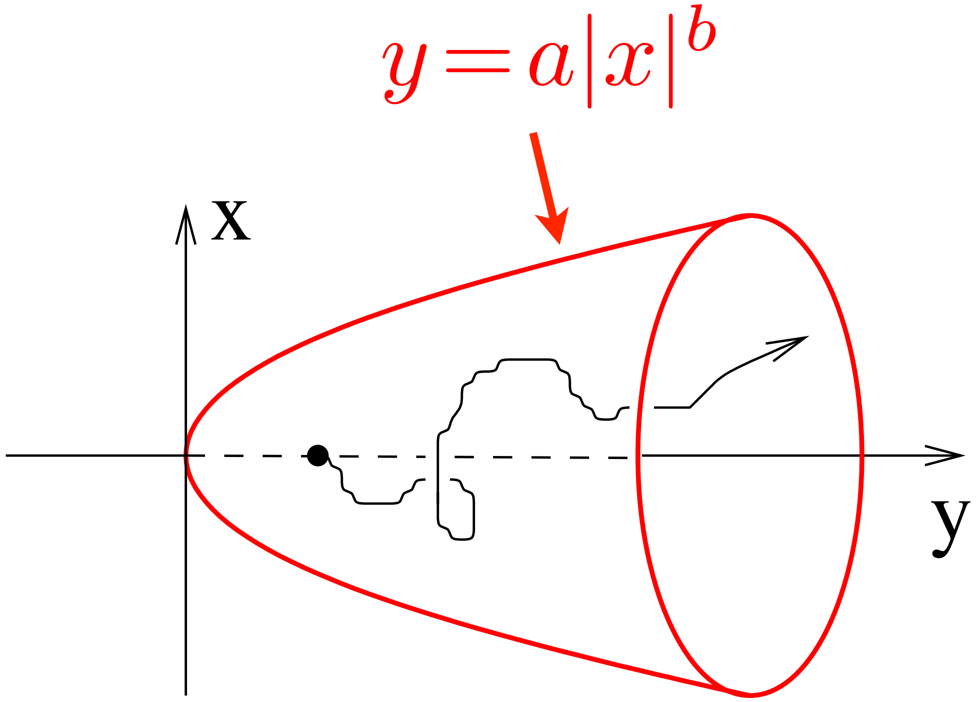


Survival in Wedges, Cones, and Paraboloids

Bañuelos et al (2001)
Lifshitz and Shi (2002)
KR (2010)



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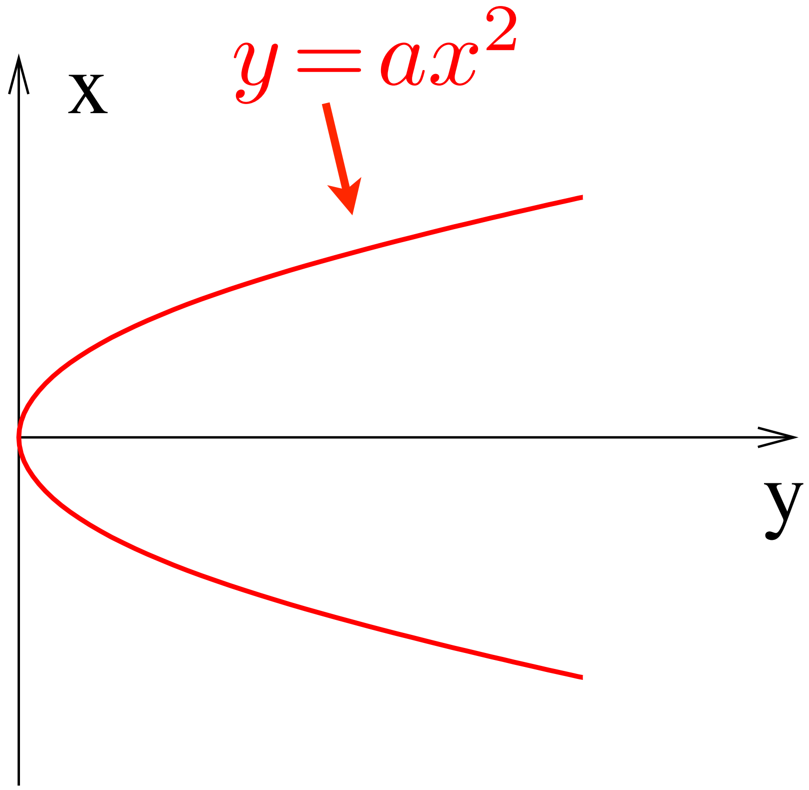

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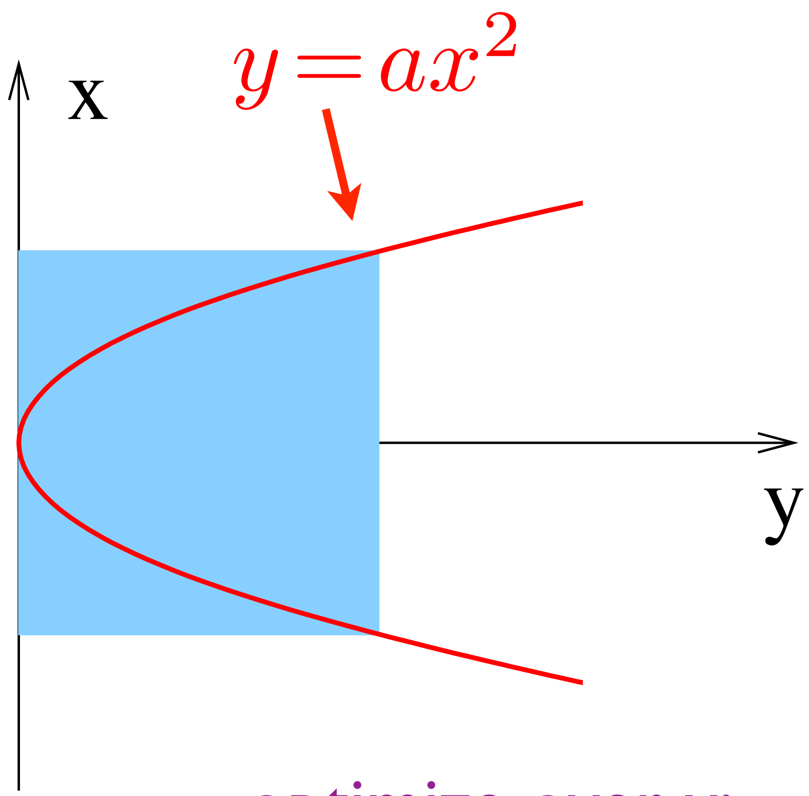


survival criterion:

prob. to remain within |x| and exit via right edge:

$$e^{-\pi^2 Dt/(2x)^2} \times e^{-y^2/4Dt}$$

optimize over y:

$$S(t) < \int_0^\infty e^{-\pi^2 Dt/(2x)^2} \times e^{-y^2/4Dt}\, dy$$

$$< \exp\left[-\tfrac{3}{4}\left(\tfrac{a\pi^2}{2}\right)^{2/3}(Dt)^{1/3}\right] \equiv \exp[-At^{1/3}]$$

$$A = \tfrac{3}{8}\pi^{4/3} \qquad A_{\rm exact} = \tfrac{3}{8}\pi^2$$

Survival in Wedges, Cones, and Paraboloids

generalized paraboloid in d dimensions:

$$y = a\left(\sqrt{x_1^2 + \ldots + x_{d-1}^2}\right)^p \qquad p > 1$$

the same approach as d=2 gives:

$$S < \exp\left[-\frac{p+1}{4}\left(\frac{4j^2_{(d-3)/2}\, a^{2/p}}{p}\right)^{\frac{p}{p+1}} (Dt)^{\frac{p-1}{p+1}}\right]$$

Hitting Times in Wedges, Cones, and Paraboloids

backward equation: $$T \equiv \langle t(\vec{r}) \rangle = \sum_{\vec{r}\,'} p_{\vec{r} \to \vec{r}\,'} [t(\vec{r}\,') + \delta t]$$

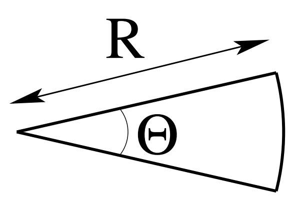
continuum limit: $$D\nabla^2 T = -1 \qquad T\big|_{\text{boundary}} = 0$$

relation to survival probability: $$T = \int_0^\infty t\, F(t)\, dt = -\int_0^\infty t\, \frac{\partial S}{\partial t}\, dt = \int_0^\infty S(t)\, dt$$

Hitting Times in Wedges, Cones, and Paraboloids

infinite wedge: if $S(t)$ decays faster than t^{-1}, then $T < \infty$

$$\Theta_c = \begin{cases} \pi/2 & d = 2 \\ 2\cos^{-1}(1/\sqrt{3}) \approx 109.47 & d = 3 \end{cases}$$

pie wedge: 

$$T(r,\theta) = \frac{r^2}{4D}\left(\frac{\cos 2\theta}{\cos\Theta} - 1\right) + \sum_{n=0}^{\infty} A_n r^{\lambda_n}\cos(\lambda_n\theta)$$

$$\lambda_n = (2n+1)\tfrac{\pi}{\Theta}$$

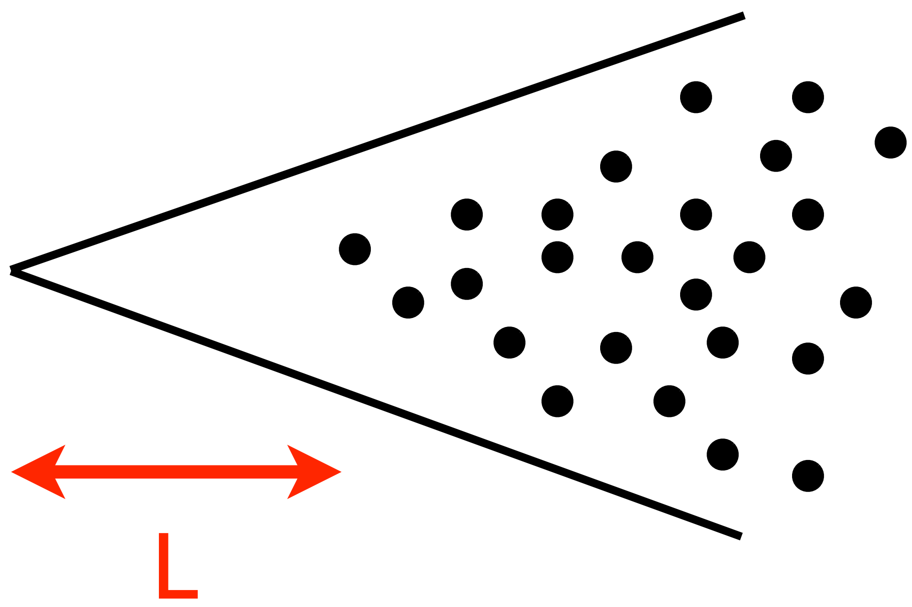
$$A_n = \frac{(-1)^{n+1}\, 4R^{2-\lambda_n}}{D\Theta\lambda_n(\lambda_n^2 - 4)}$$

if $\Theta \geq \frac{\pi}{2}$, then $T \sim r^{\pi/\Theta} R^{2-\pi/\Theta}$

divergent for $\Theta > \pi/2$

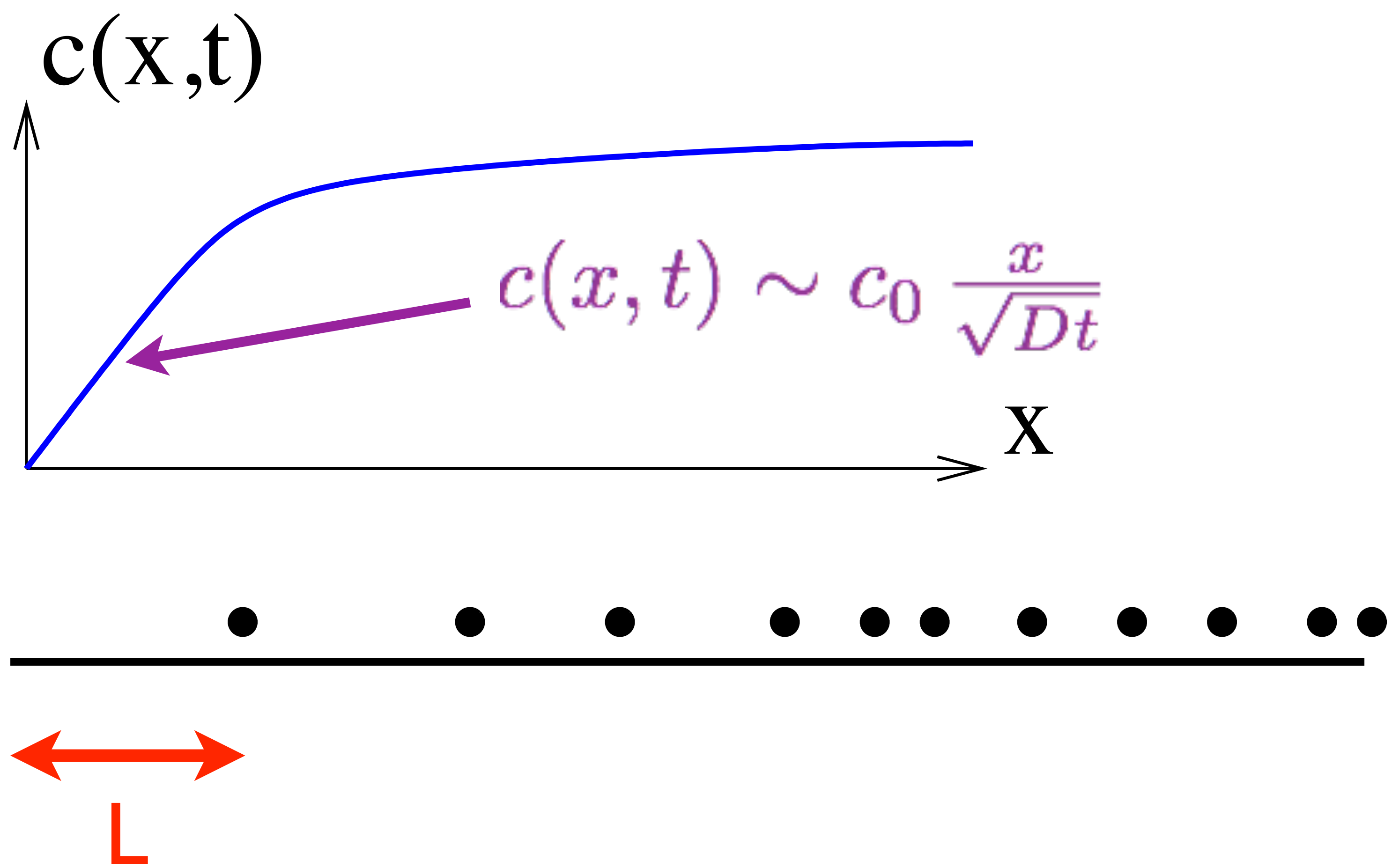
parabola: $T = \frac{1}{2D}(y - x^2)$

The Closest Particle



The Closest Particle

warm-up: 1d

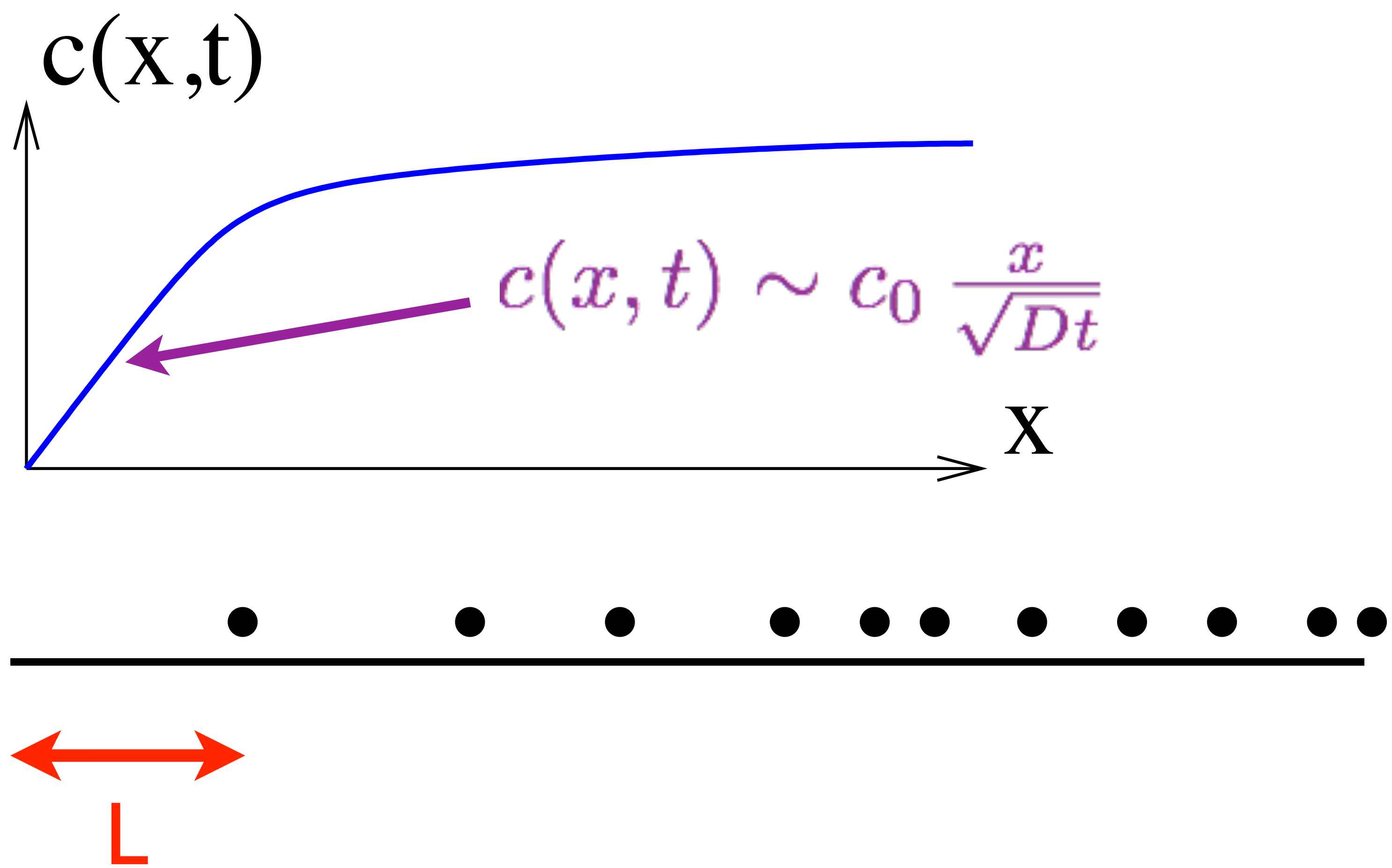


closest particle criterion:

$$\int_0^L c(x,t)\,dx = 1$$

The Closest Particle

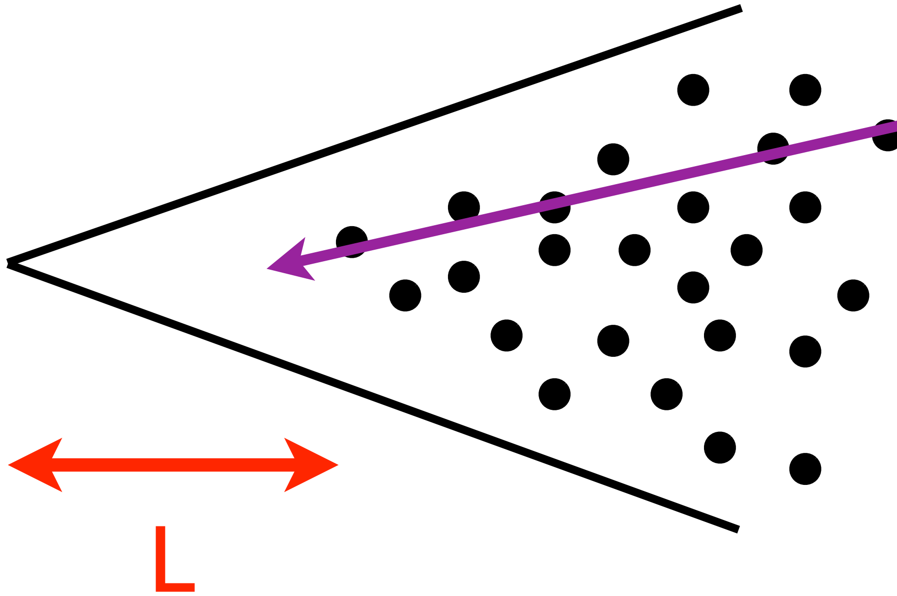
warm-up: 1d



closest particle criterion:

$$\int_0^L c(x,t)\,dx = 1 \rightarrow L \sim \left(\frac{Dt}{c_0^2}\right)^{1/4}$$

The Closest Particle



$c(r,t) \sim c_0 \left(\frac{r}{\sqrt{Dt}}\right)^{-\pi/\Theta} f(\theta)$

$\int_0^L r\, c(r,t) dr \int_0^\Theta g(\theta) d\theta = 1 \rightarrow L \sim (Dt)^{\pi/(2\pi+4\Theta)}\; c_0^{-2\Theta/(\pi+2\Theta)}$

$$L \sim \begin{cases} (Dt)^{1/2} & \Theta \downarrow 0 \\ (Dt)^{1/4}\, \rho^{-1/4} & \Theta = \pi/2 \\ (Dt)^{1/6}\, \rho^{-1/3} & \Theta = \pi \\ (Dt)^{1/10}\, \rho^{-2/5} & \Theta = 2\pi \end{cases}$$

crack

corner

plane

needle

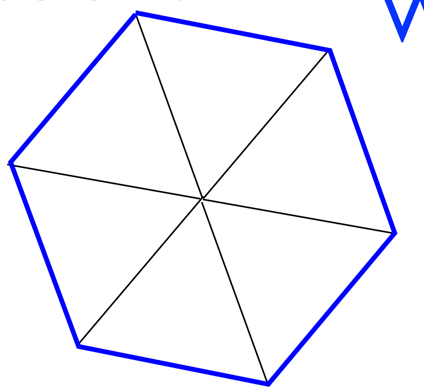
Some Concluding Remarks

- particle capture problems most interesting in 1d
- 3-particle problems well understood:

no reversal, $123 \not\Rightarrow 321$	$t^{-3/10}$
3 never trails, $123 \not\Rightarrow 321, 312$	$t^{-3/8}$
3 always leads 1	$t^{-3/6}$
3 always leads 1 & 2	$t^{-3/4}$
order preserved	$t^{-3/2}$

- several-particle problems partially understood:

Weyl chamber



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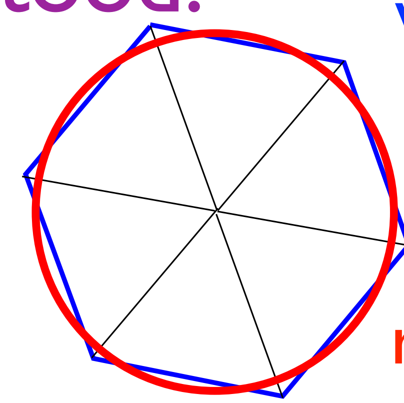
- several-particle problems partially understood:

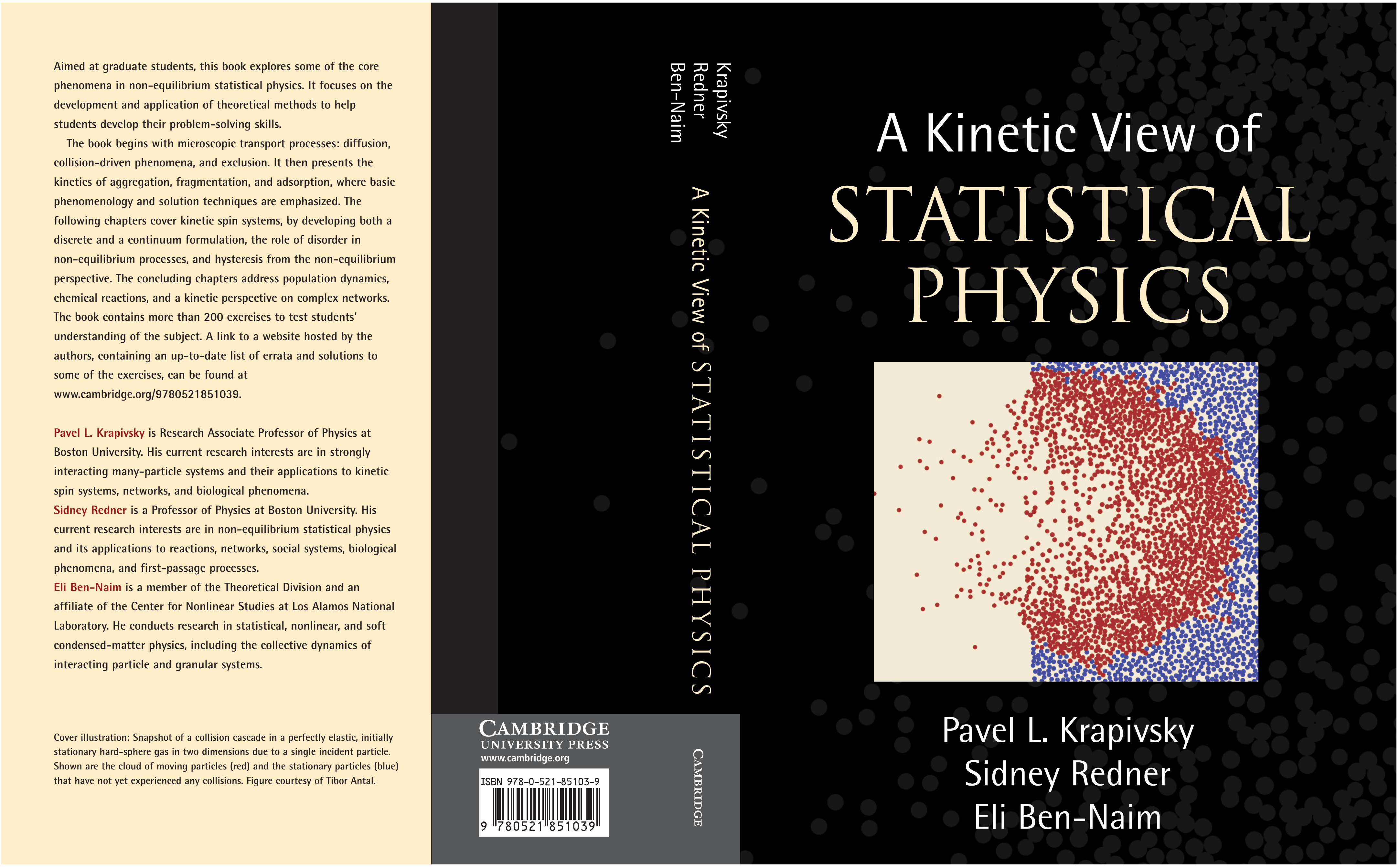
  accurate but uncontrolled approximation for survival probability



Ben-Naim & Krapivsky (2010)

- $N \rightarrow \infty$ approximation simple, powerful; cheap "derivation" of iterated logarithm law

Crass Commercialism



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1. Aperitifs
2. Diffusion
3. Collisions
4. Exclusion
5. Aggregation
6. Fragmentation
7. Adsorption
8. Spin Dynamics
9. Coarsening
10. Disorder
11. Hysteresis
12. Population Dynamics
13. Diffusion Reactions
14. Complex Networks

> 200 problems