

Diffusing Predators Hunting a Diffusing Prey

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Basic question: What is the survival probability of a diffusing prey that is hunted by diffusing predators?

Fact: one dimension most interesting $S_N(t) \sim t^{-\beta_N}$

- Outline:**
- 1 & 2 predators *exactly soluble*
 - 3 predators *accurate exponent by electrostatics*
 - $N \gg 1$ predators $\beta_N \approx \frac{1}{4} \ln N$
 - $N = \infty$ predators $S(t) \approx \exp[-\frac{1}{8} \ln^2(t)]$
 - simple argument for iterated logarithm law
 - survival in wedges, cones, paraboloids

Dimension Dependence

$d > 2$: hunt *unsuccessful*; the prey may survive forever

Polya (1921), Bramson & Griffeath (1991)

$d = 2$: hunt *successful*, but hunters still essentially independent

$$\rightarrow S_N(t) \sim [S_1(t)]^N$$

$d = 1$: *successful* hunt $S_N(t) \sim t^{-\beta_N}$

- *surrounded prey*: adding hunters is efficient $\rightarrow \beta_N \sim N$
- *chased prey*: adding hunters is inefficient \rightarrow **slow decay**
 β_N sublinear in N

effective correlation between hunters

One Diffusing Hunter, One Stationary Prey



probability distribution of the prey-hunter separation:

$$p(x, t) = \frac{1}{\sqrt{4\pi Dt}} \left[e^{-(x-x_0)^2/4Dt} - e^{-(x+x_0)^2/4Dt} \right]$$

$x_0 \equiv$ initial separation of hunter and prey

$x \equiv$ current separation of hunter and prey

survival probability; integrate over all x :

$$S_1(t) = \text{erf} \left(\frac{x_0}{\sqrt{4Dt}} \right) \sim \frac{x_0}{\sqrt{\pi Dt}} \quad \text{as } t \rightarrow \infty$$

$$S_1 \sim t^{-\beta_1} \quad \text{with} \quad \beta_1 = \frac{1}{2}$$

One Diffusing Hunter, One Diffusing Prey



x_1

<



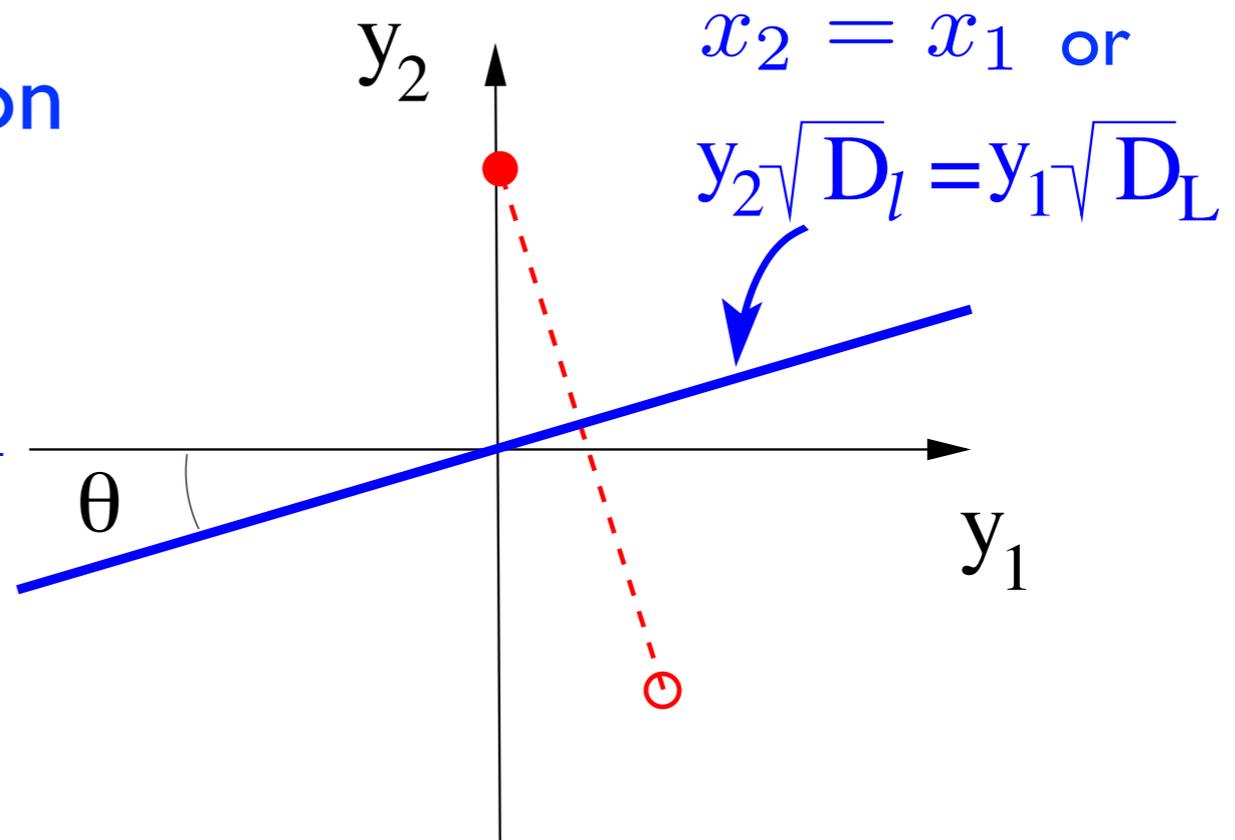
x_2

map to *isotropic* 2d diffusion

$$y_1 = x_1 / \sqrt{D_L}$$

$$y_2 = x_2 / \sqrt{D_\ell}$$

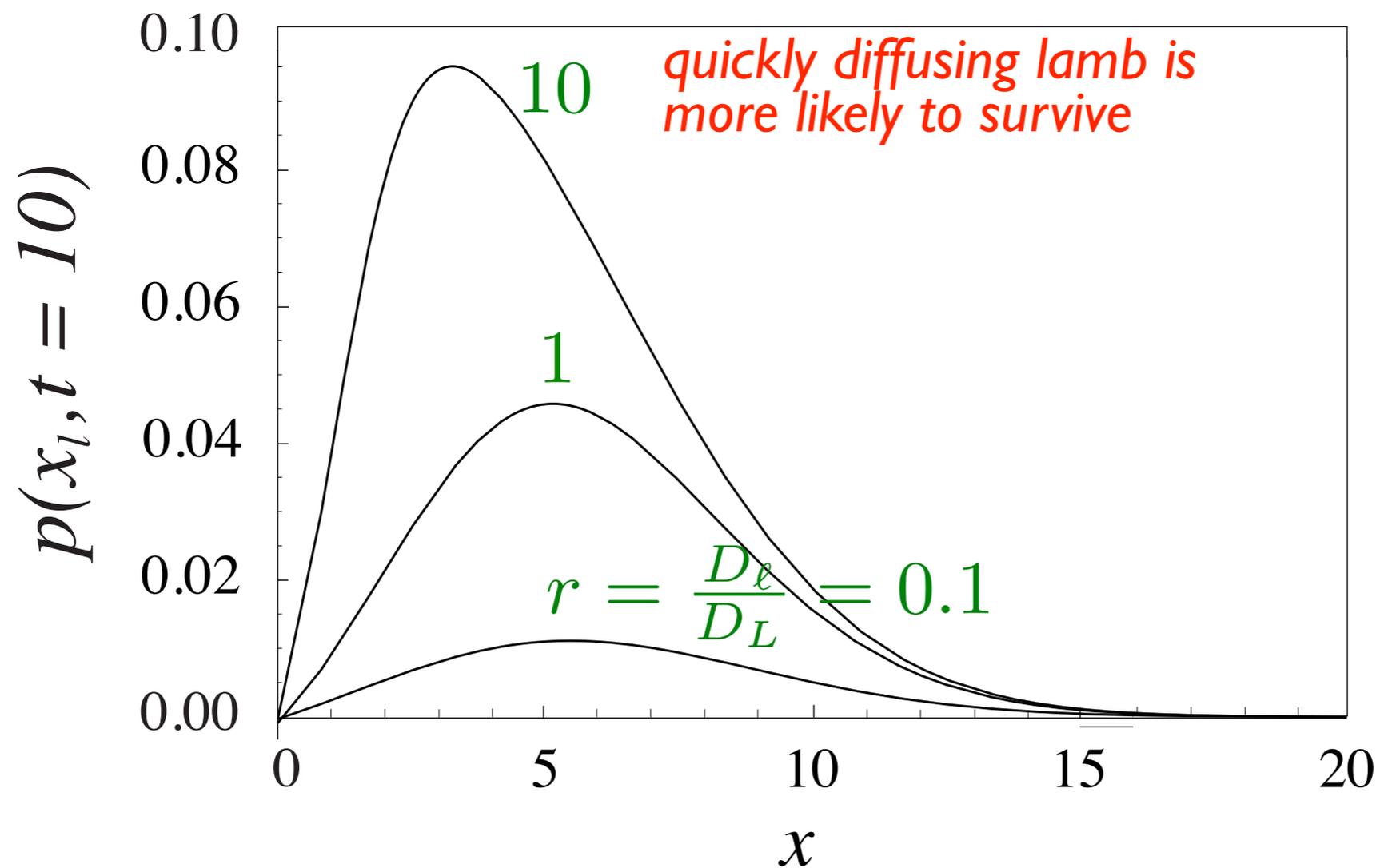
$$\theta = \tan^{-1} \sqrt{\frac{D_L}{D_\ell}}$$



$$p(y_1, y_2, t) = \frac{1}{4\pi t} \left[e^{-[y_1^2 + (y_2 - \sqrt{D_\ell})^2]/4t} - e^{-[(y_1 - \sqrt{D_\ell} \sin 2\theta)^2 + (y_2 + \sqrt{D_\ell} \cos 2\theta)^2]/4t} \right]$$

Prey Probability Distribution

$$p(x_\ell, t) = \frac{1}{\sqrt{16\pi D_\ell t}} \left[e^{-(x_\ell - 1)^2 / 4D_\ell t} \operatorname{erfc} \left(-\frac{x_\ell \cot \theta}{\sqrt{4D_\ell t}} \right) - e^{-(x_\ell + \cos 2\theta)^2 / 4D_\ell t} \operatorname{erfc} \left(\frac{\sin 2\theta - x_\ell \cot \theta}{\sqrt{4D_\ell t}} \right) \right]$$



Two Diffusing Hunters, One Diffusing Prey



x_1



x_2



x_3

require $x_2 < x_3$ *and* $x_1 < x_3$

map to 3d diffusion

Two Diffusing Hunters, One Diffusing Prey



x_1



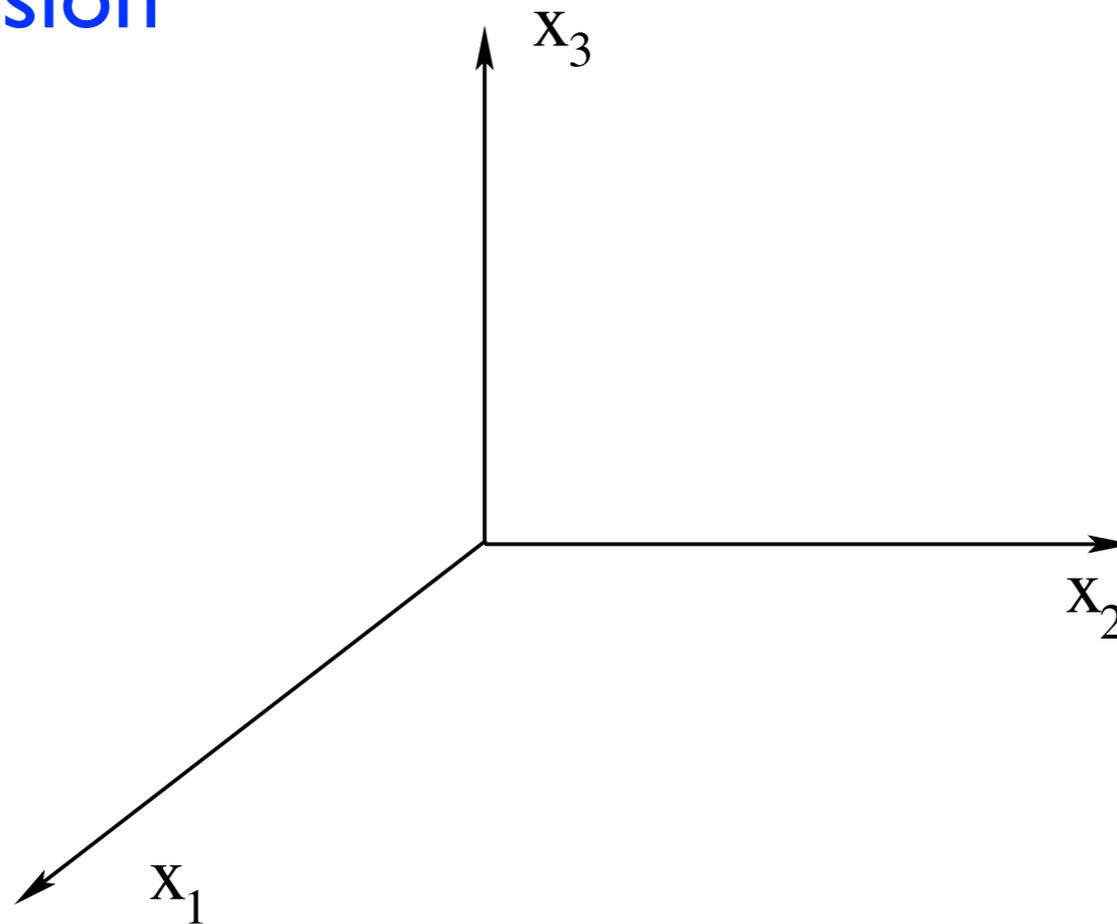
x_2



x_3

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Two Diffusing Hunters, One Diffusing Prey



x_1



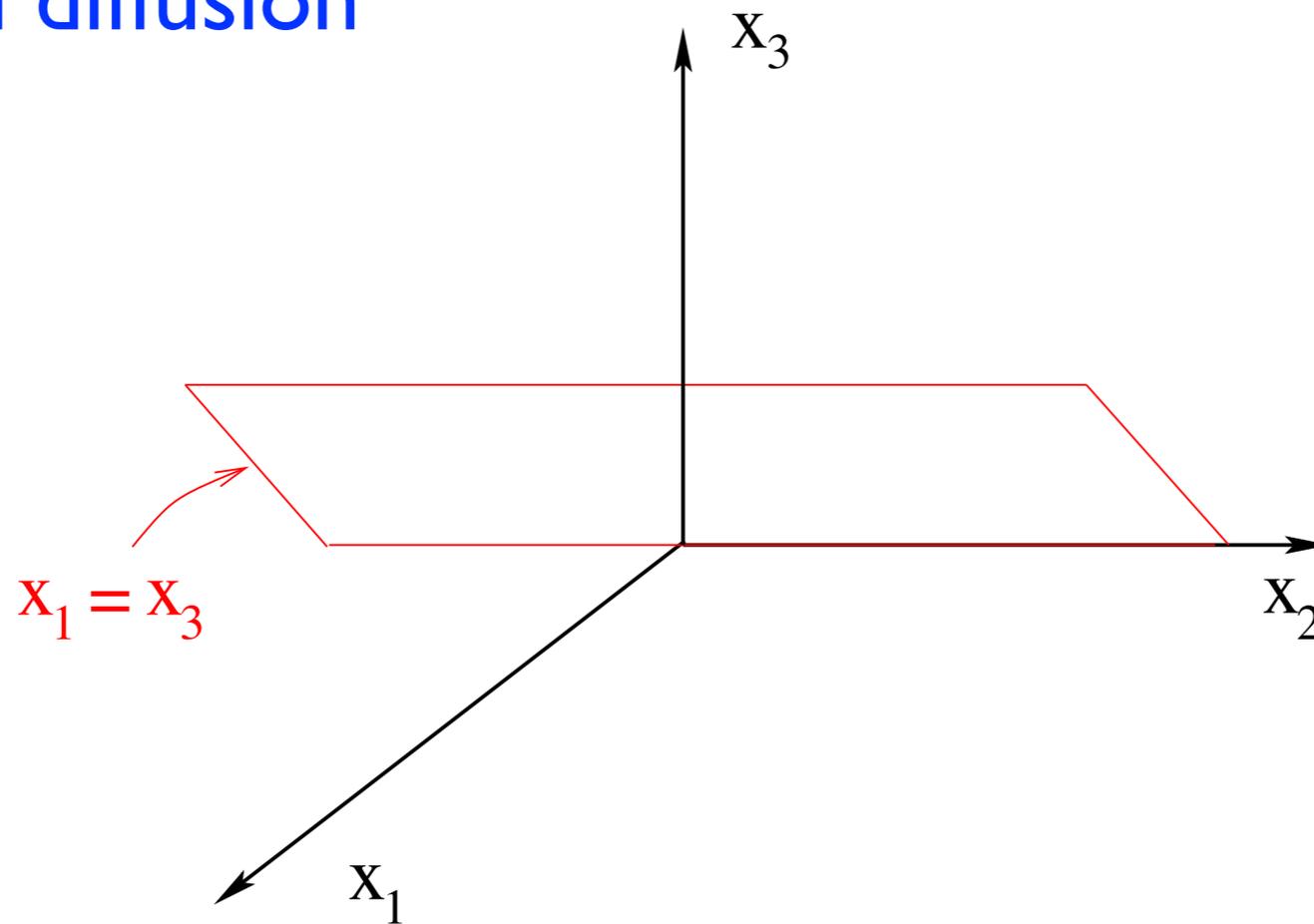
x_2



x_3

require $x_2 < x_3$ *and* $x_1 < x_3$

map to 3d diffusion



Two Diffusing Hunters, One Diffusing Prey



x_1



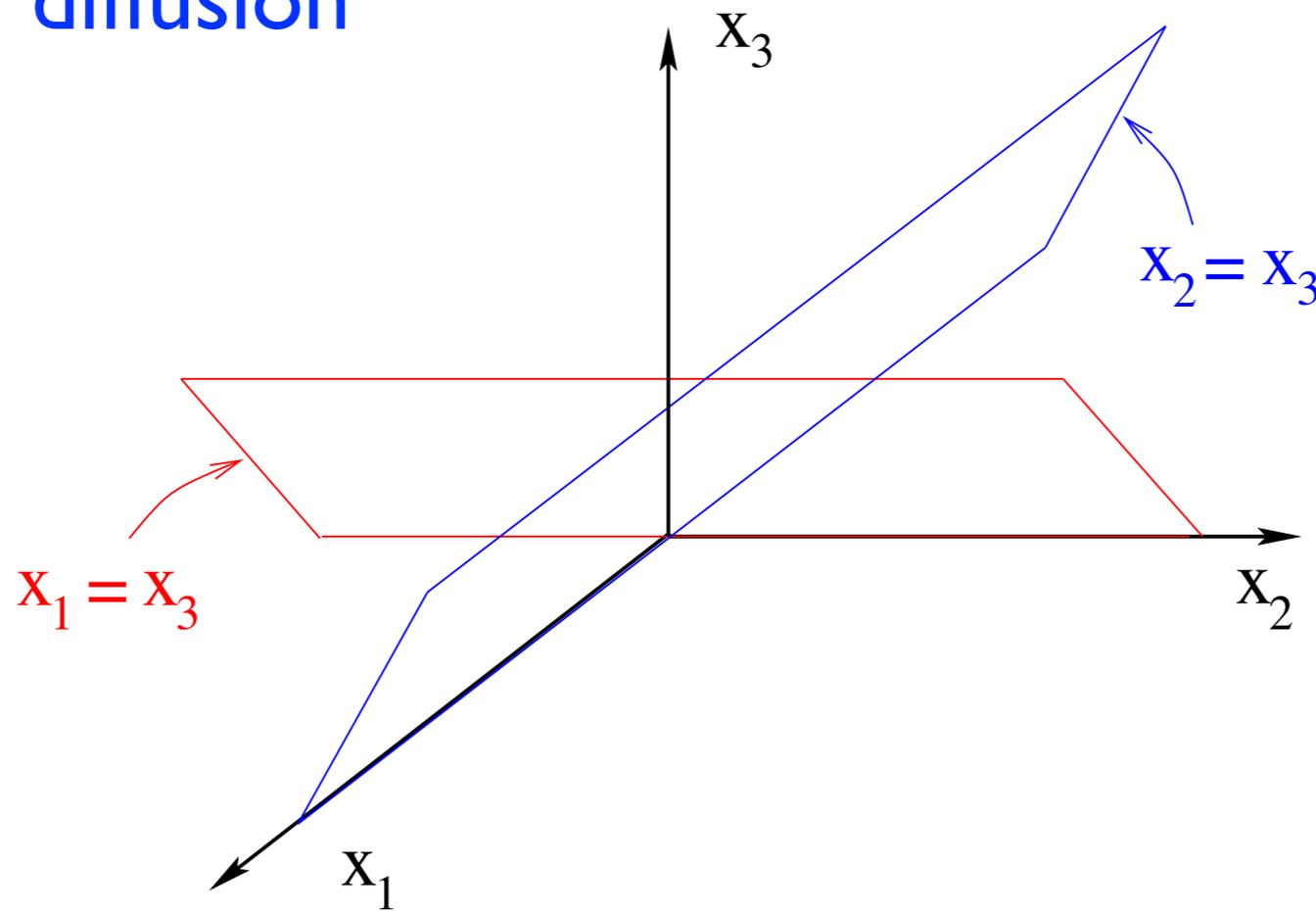
x_2



x_3

require $x_2 < x_3$ *and* $x_1 < x_3$

map to 3d diffusion



Two Diffusing Hunters, One Diffusing Prey



x_1



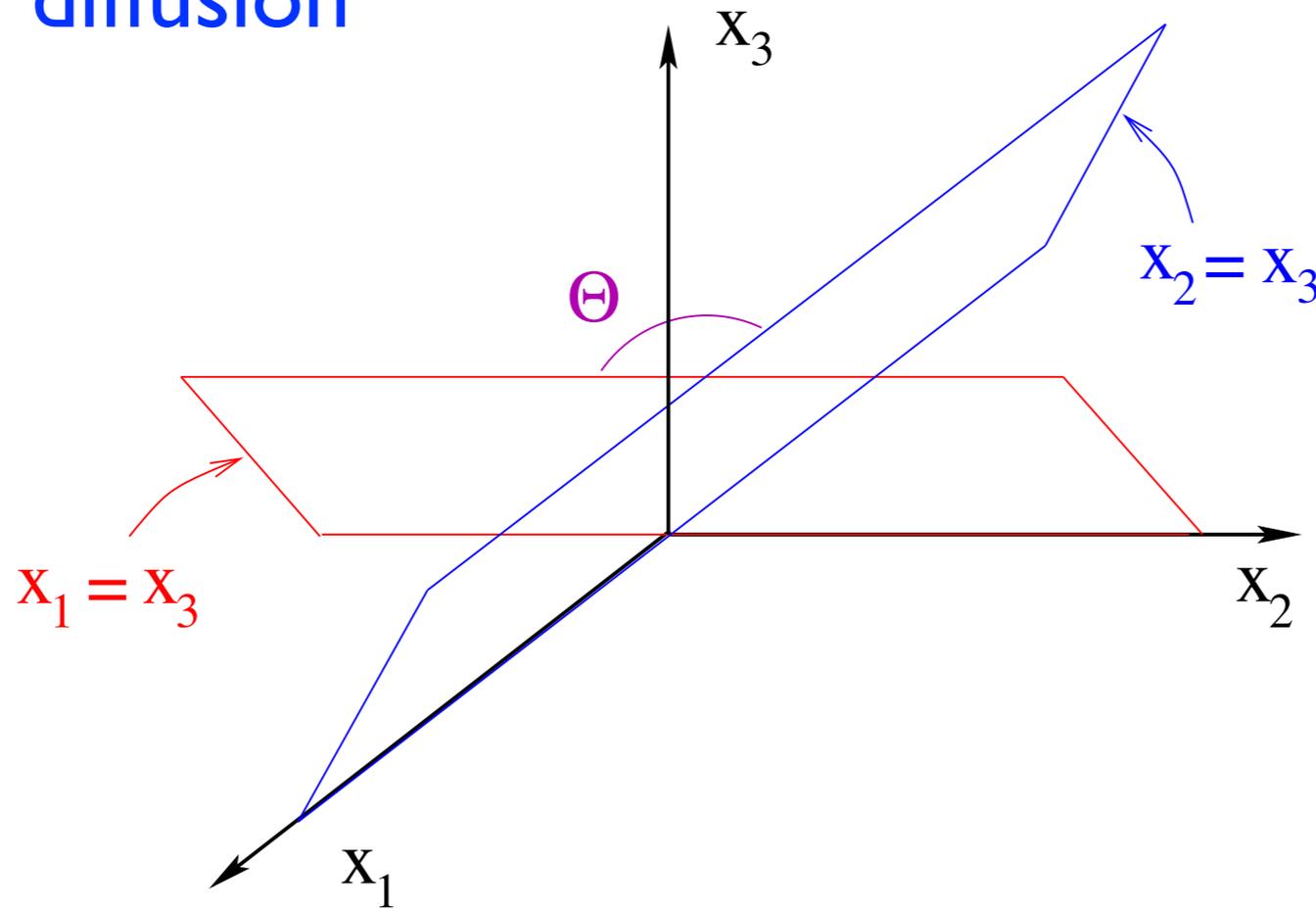
x_2



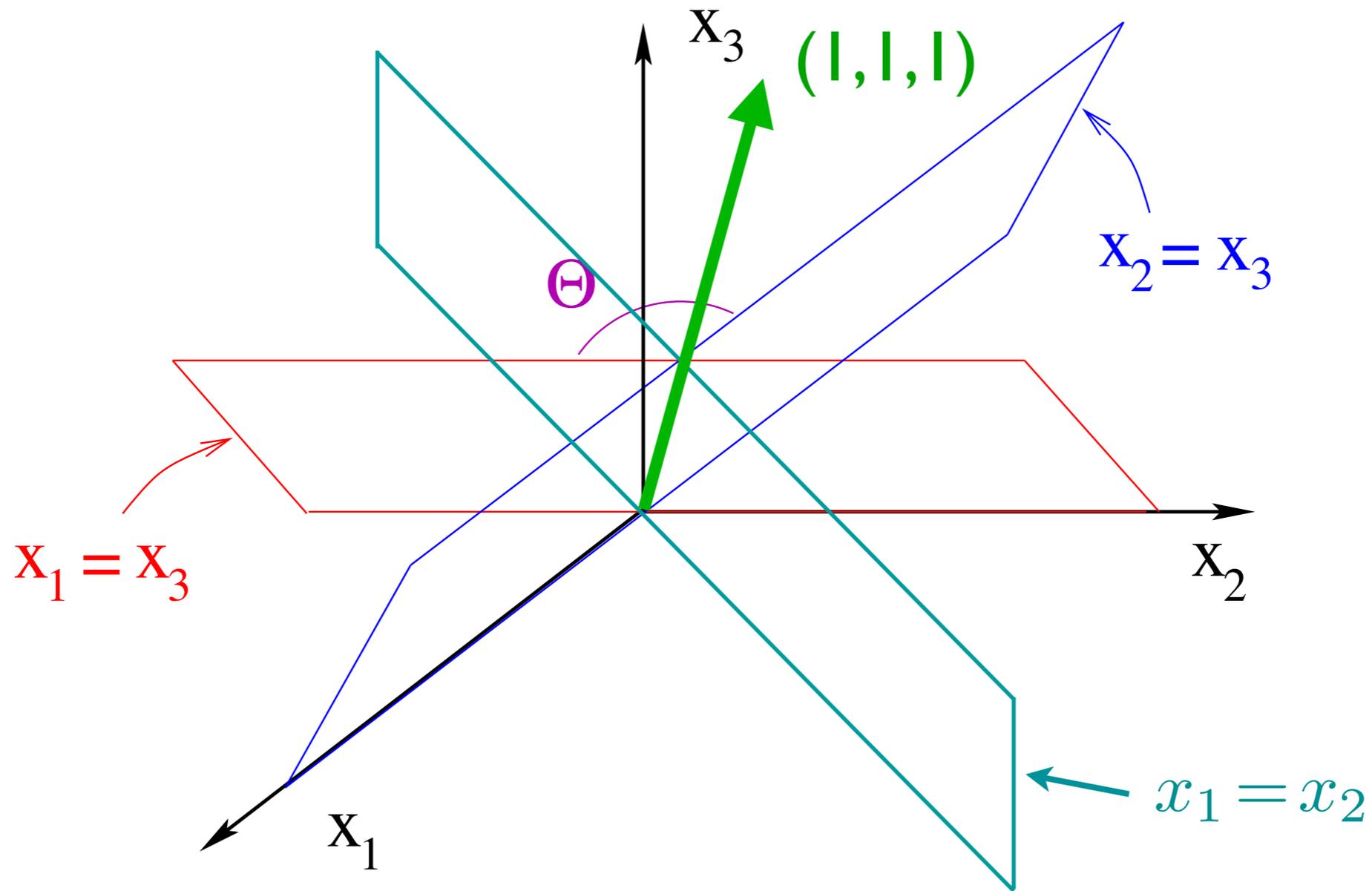
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map to 3d diffusion

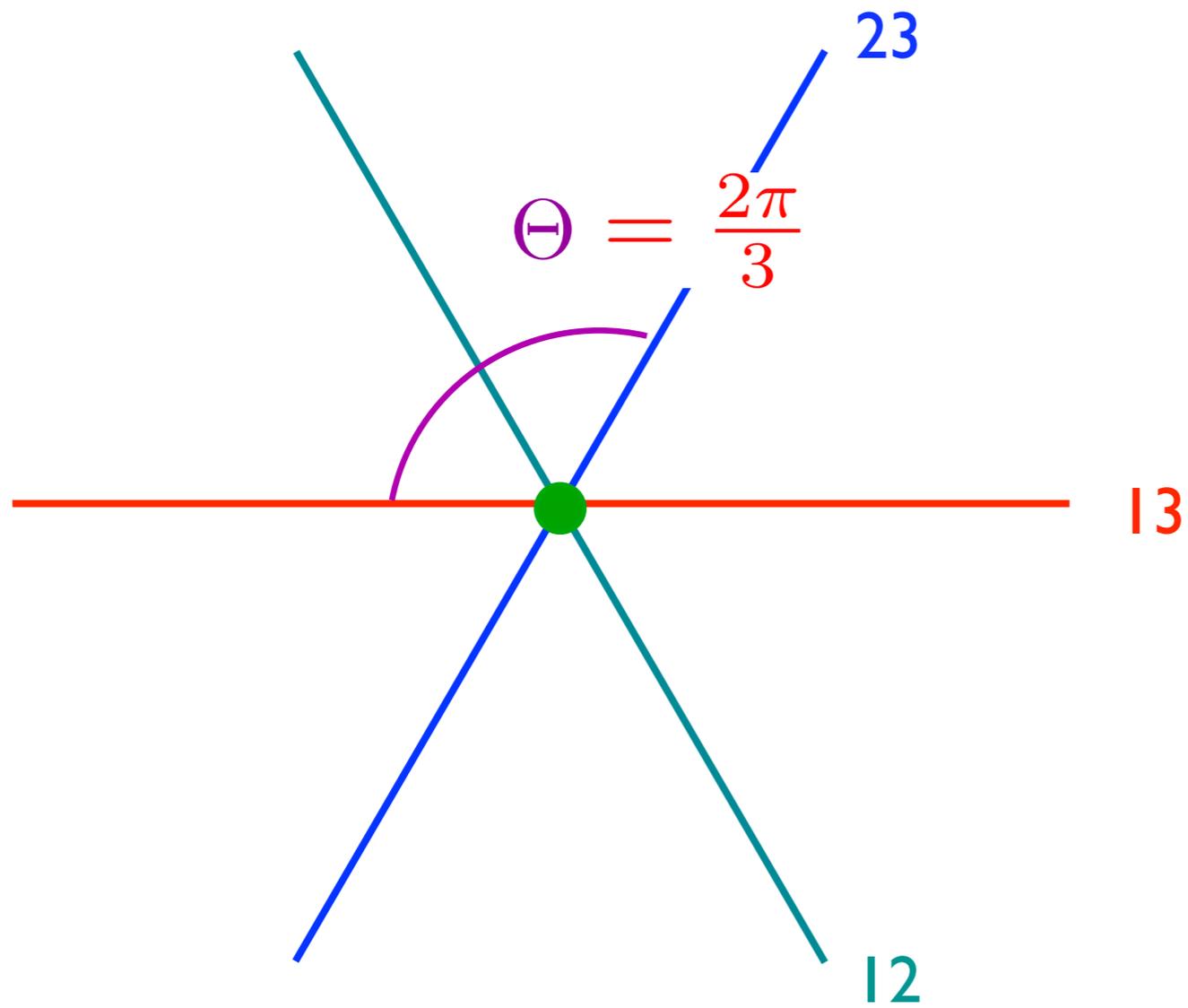


Equivalence to Diffusion in Absorbing Wedge

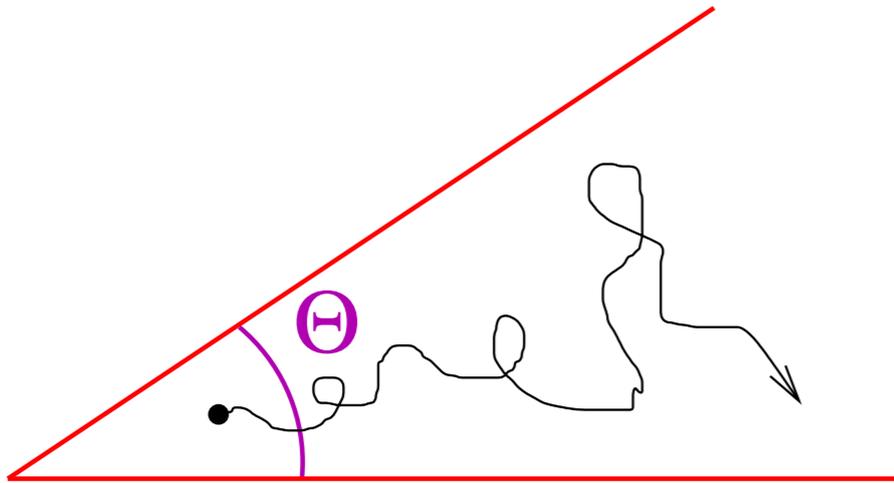


Equivalence to Diffusion in Absorbing Wedge

view perpendicular to (1,1,1)



Diffusion in Absorbing Wedge



$$S(t) \sim t^{-\pi/2\Theta}$$

Carlsaw & Jaeger (1959)

electrostatic
equivalence

$$\phi(r) \sim r^{-\pi/\Theta}$$

equivalent to injecting
diffusing particles at
fixed rate

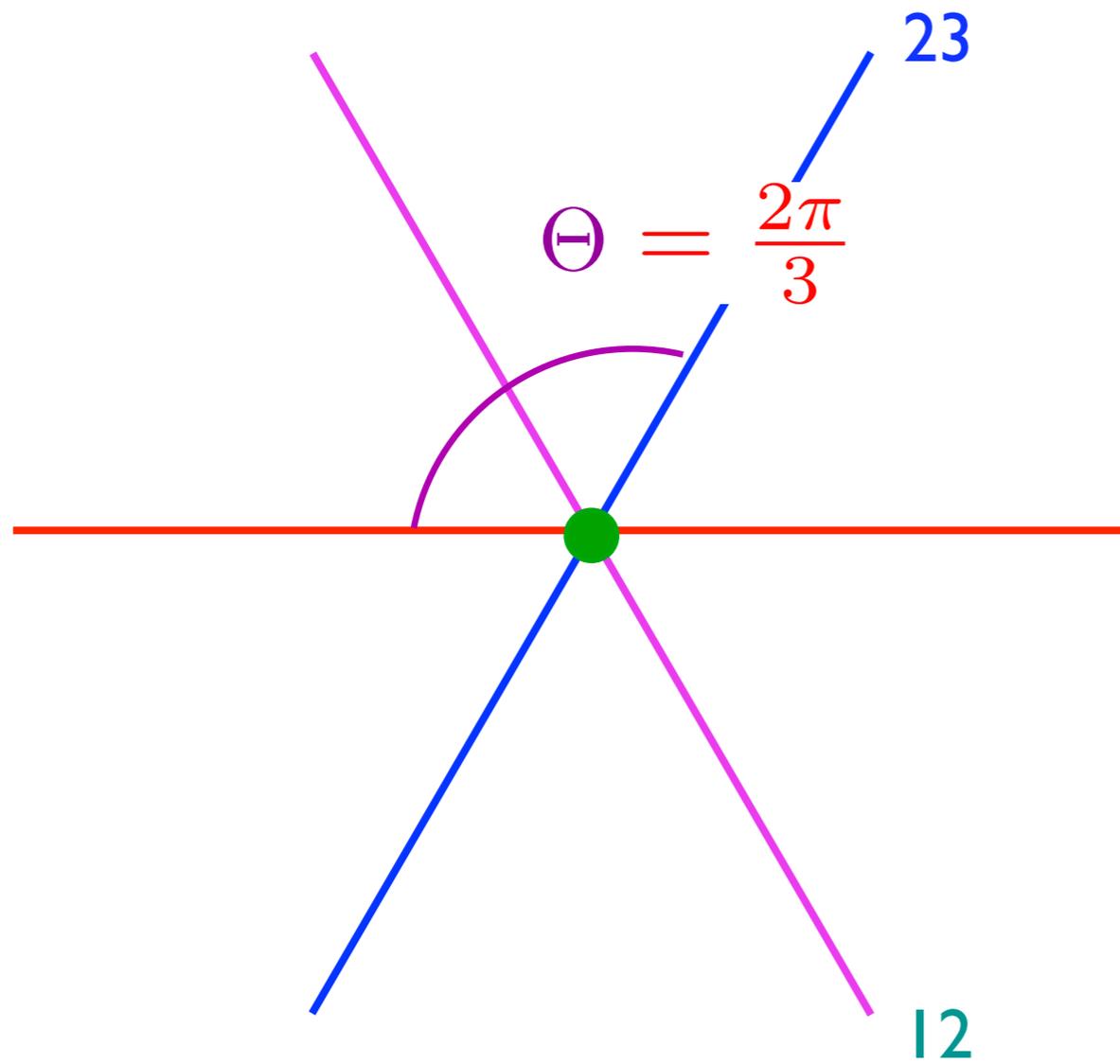
number of
particles within
wedge

$$N(t) \sim \int_0^\Theta d\theta \int_0^{\sqrt{Dt}} \phi(r) r dr$$

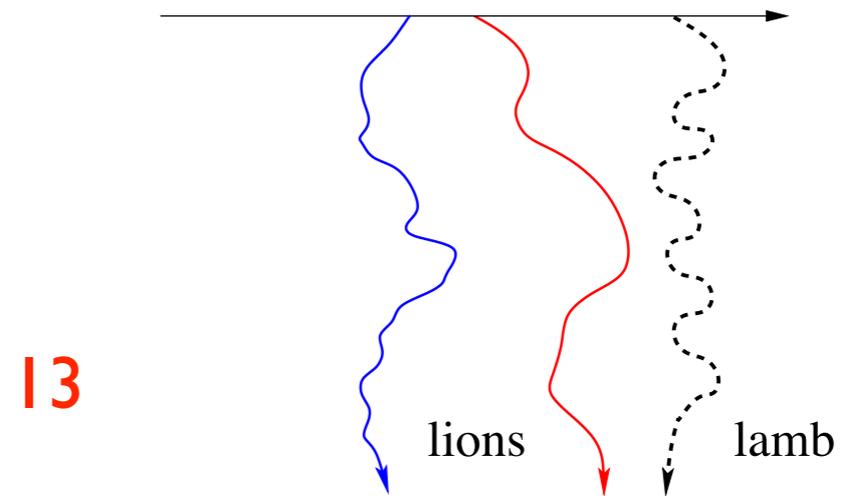
$$\sim \int_0^{\sqrt{Dt}} r^{1-\pi/\Theta} dr$$

$$\sim t^{1-\pi/2\Theta}$$

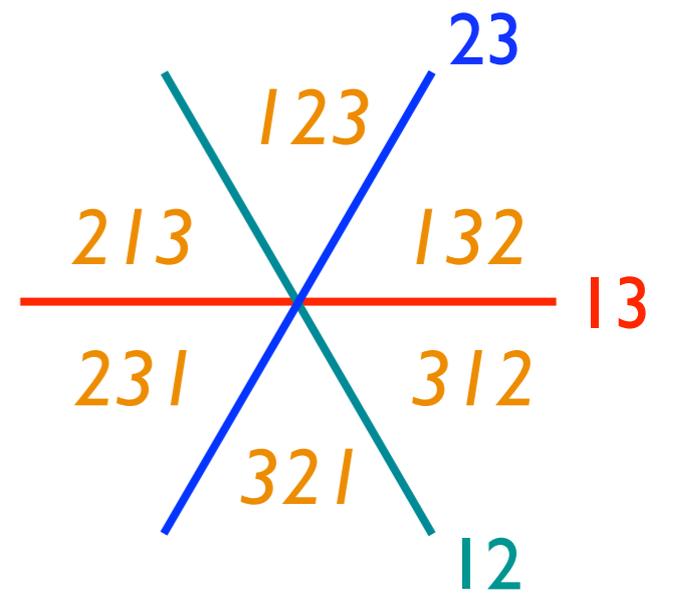
Equivalence to Diffusion in Absorbing Wedge



$$S_2(t) \sim t^{-3/4} \quad \text{chased prey}$$



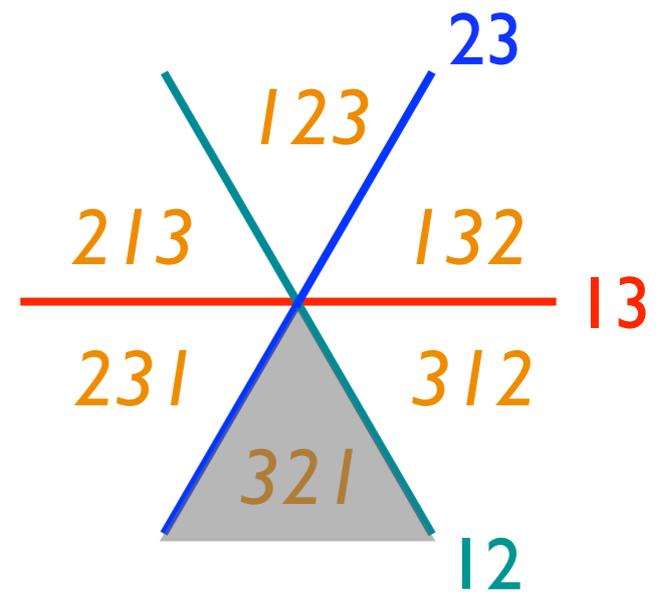
Systematics of the Equivalence



Systematics of the Equivalence

no reversal, $123 \not\Rightarrow 321$

$t^{-3/10}$



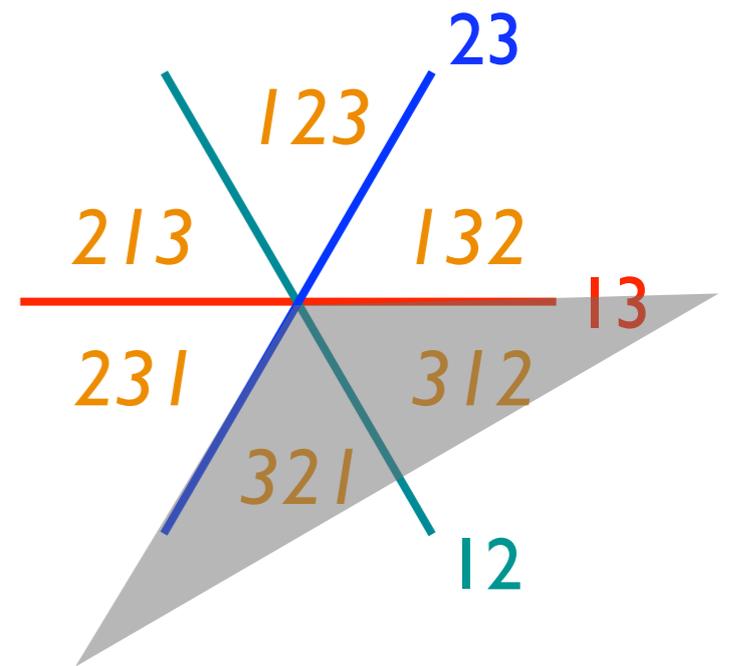
Systematics of the Equivalence

no reversal, $123 \not\Rightarrow 321$

$$t^{-3/10}$$

3 never trails, $123 \not\Rightarrow 321, 312$

$$t^{-3/8}$$



Systematics of the Equivalence

no reversal, $123 \not\Rightarrow 321$

3 never trails, $123 \not\Rightarrow 321, 312$

3 always leads 1

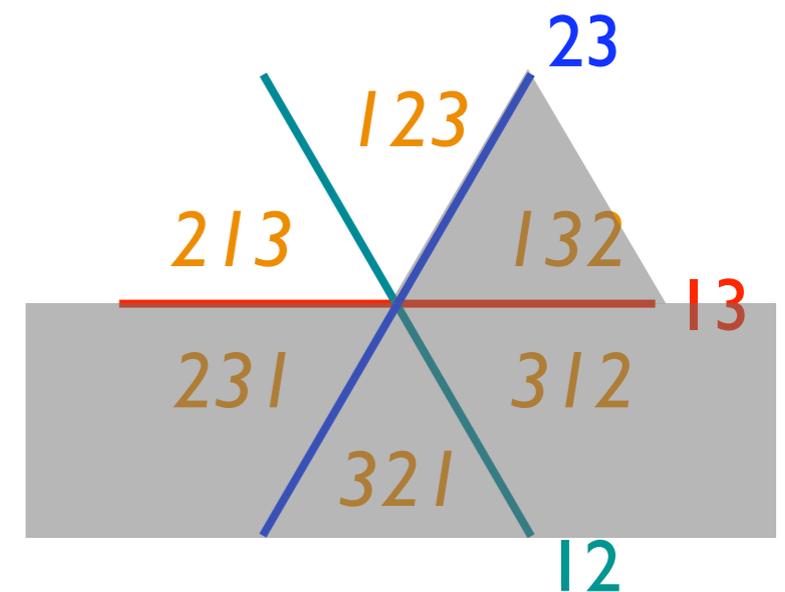
3 always leads 1 & 2

$$t^{-3/10}$$

$$t^{-3/8}$$

$$t^{-3/6}$$

$$t^{-3/4}$$



Systematics of the Equivalence

no reversal, $123 \not\Rightarrow 321$

3 never trails, $123 \not\Rightarrow 321, 312$

3 always leads 1

3 always leads 1 & 2

order preserved

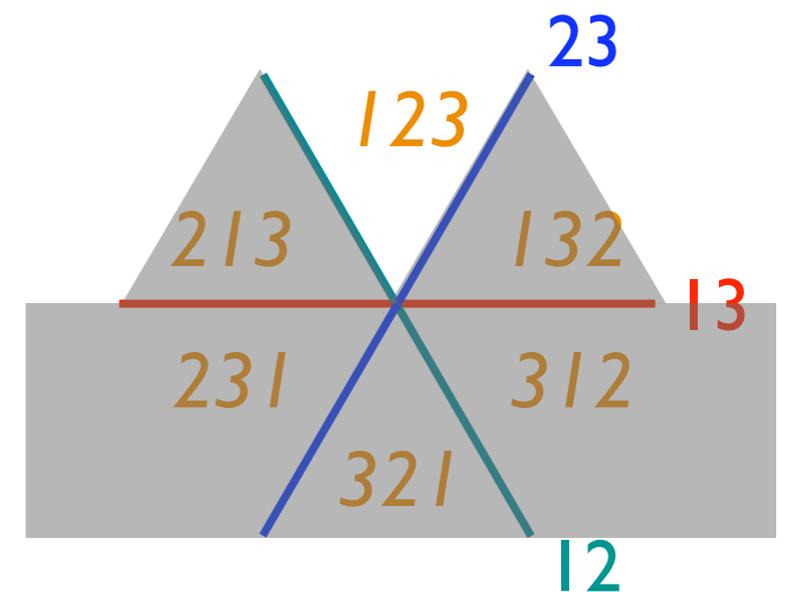
$$t^{-3/10}$$

$$t^{-3/8}$$

$$t^{-3/6}$$

$$t^{-3/4}$$

$$t^{-3/2}$$



Three Diffusing Hunters, One Diffusing Prey



x_1



x_2



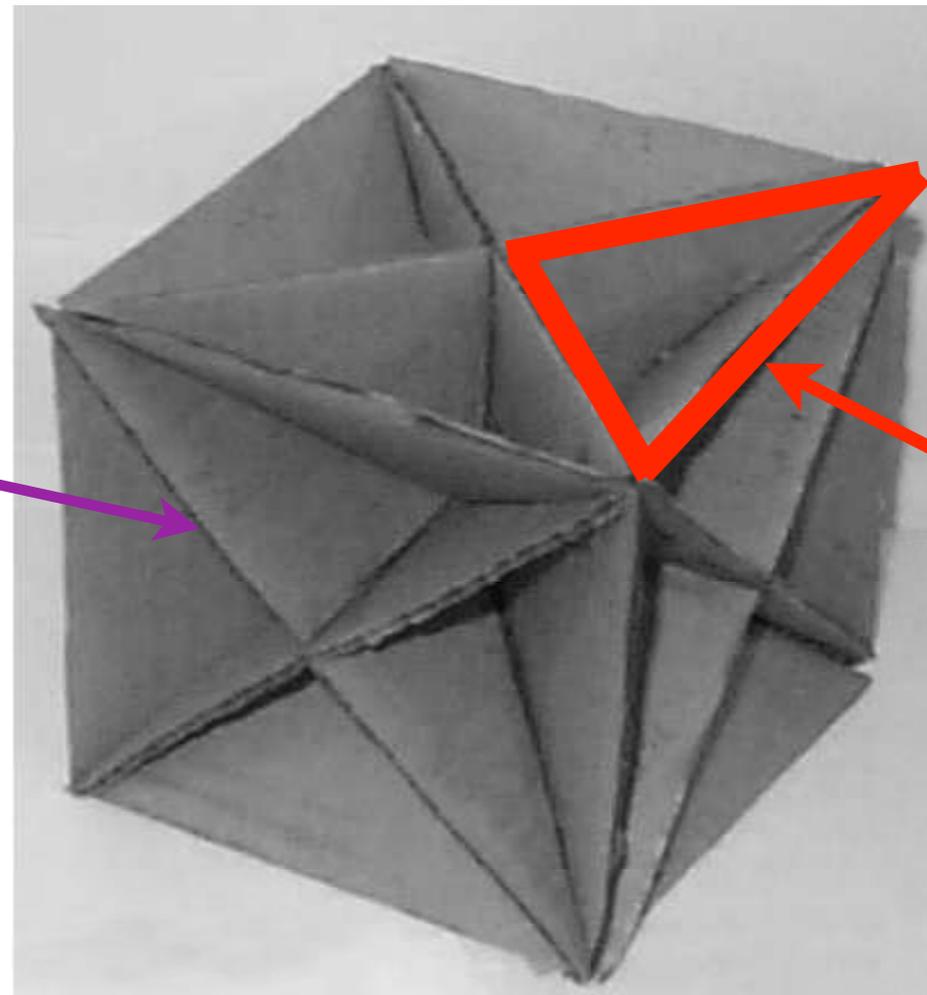
x_3



x_4

require $x_1 < x_4$, $x_2 < x_4$, $x_3 < x_4$

projection of 4-space onto 3d hyperplane \perp to $(1,1,1,1)$



each plane represents

$$x_i = x_j$$

construction by
D. ben-Avraham

each **Weyl chamber**
represents one specific
ordering; for vicious
random walks
 $x_1 < x_2 < x_3 < x_4$

Three Diffusing Hunters, One Diffusing Prey



x_1



x_2



x_3



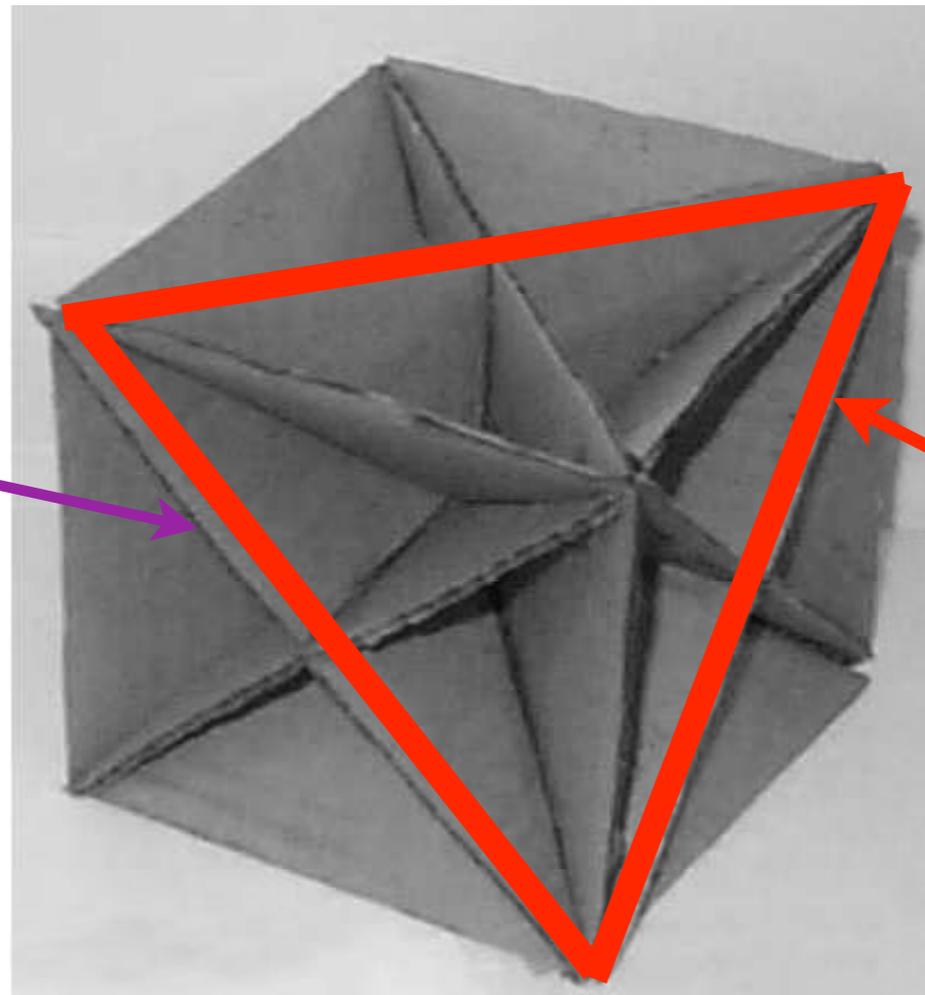
x_4

require $x_1 < x_4$, $x_2 < x_4$, $x_3 < x_4$

projection of 4-space onto 3d hyperplane \perp to $(1,1,1,1)$

each plane represents

$$x_i = x_j$$



construction by
D. ben-Avraham

Weyl chamber for
lamb survival:

$$x_1, x_2, x_3 < x_4$$

Three Diffusing Hunters, One Diffusing Prey



x_1



x_2



x_3



x_4

require $x_1 < x_4, x_2 < x_4, x_3 < x_4$

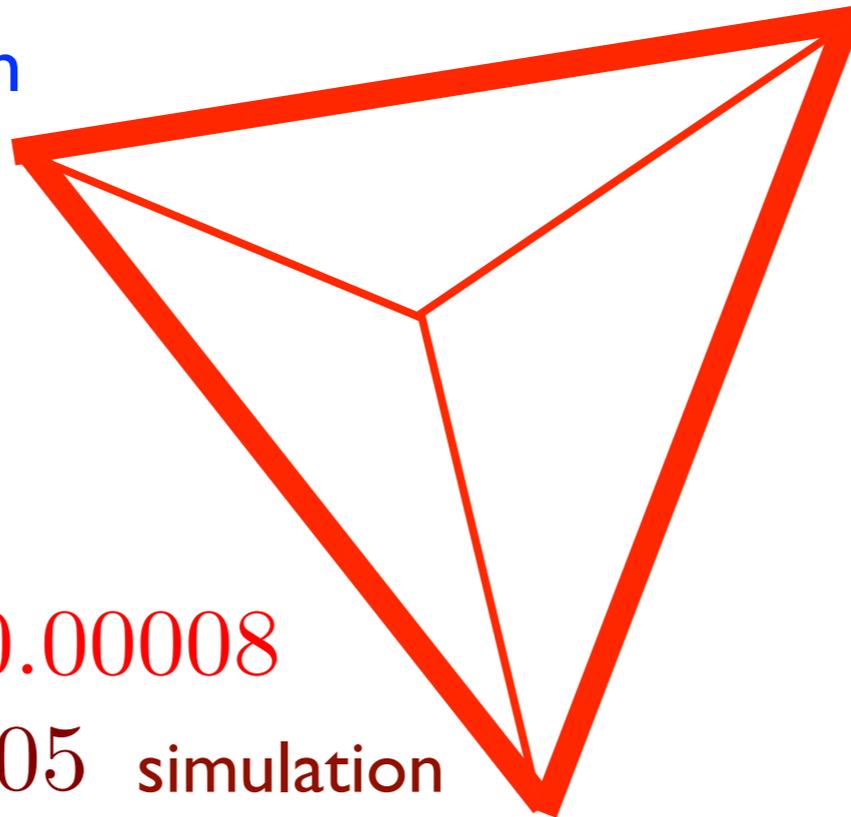
potential in Weyl chamber: $\phi(r) \sim r^{-\mu}$ $\mu = 2.82684 \pm 0.00016$
(ben-Avraham et al. 2003)

electrostatics \leftrightarrow diffusion

$$\beta = \frac{1}{2}(\mu - N + 2)$$

$$\rightarrow \beta_3 = 0.91342 \pm 0.00008$$

0.913 \pm 0.005 simulation



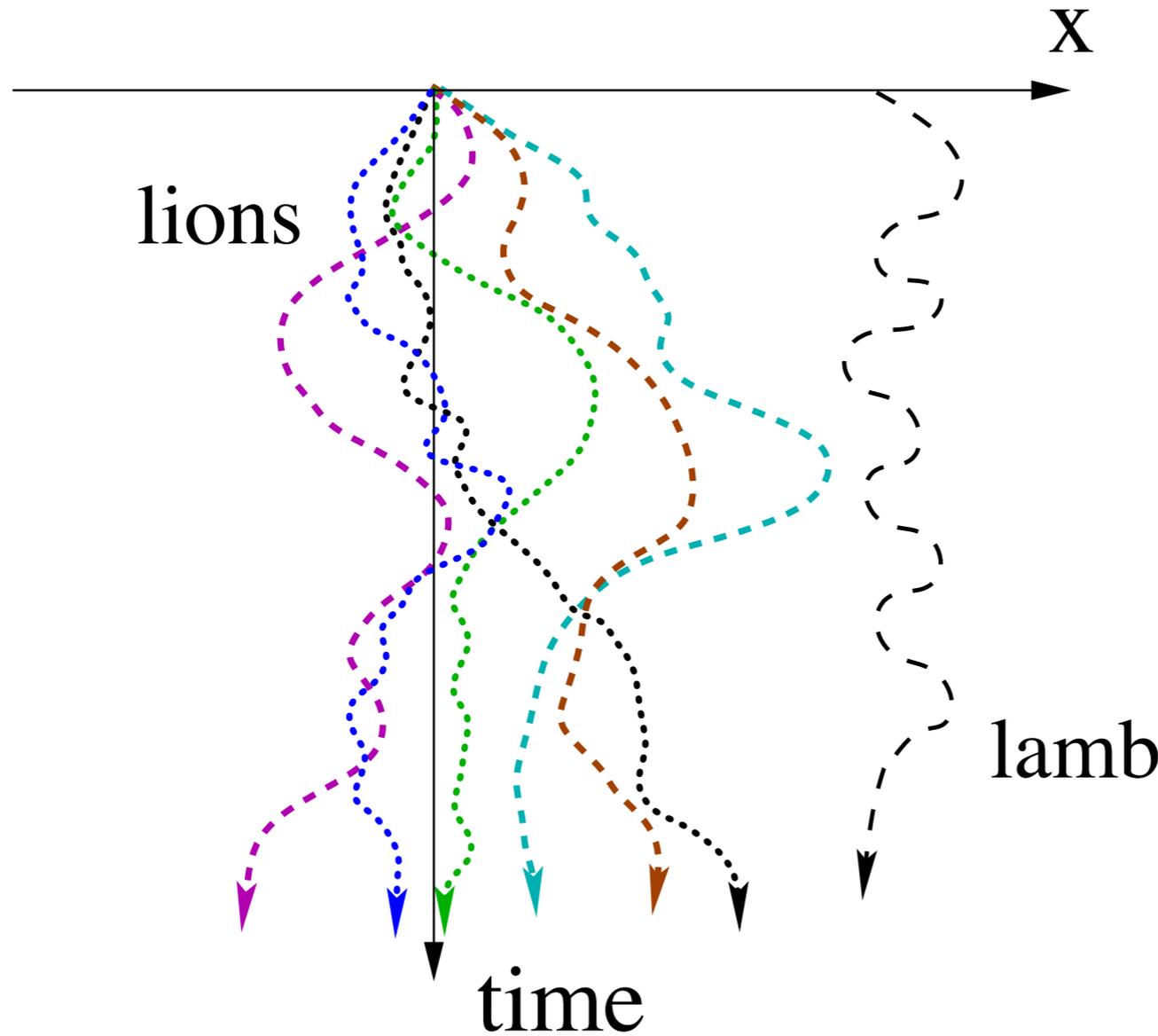
Many Diffusing Hunters, One Diffusing Prey

Krapivsky & SR (96, 99)



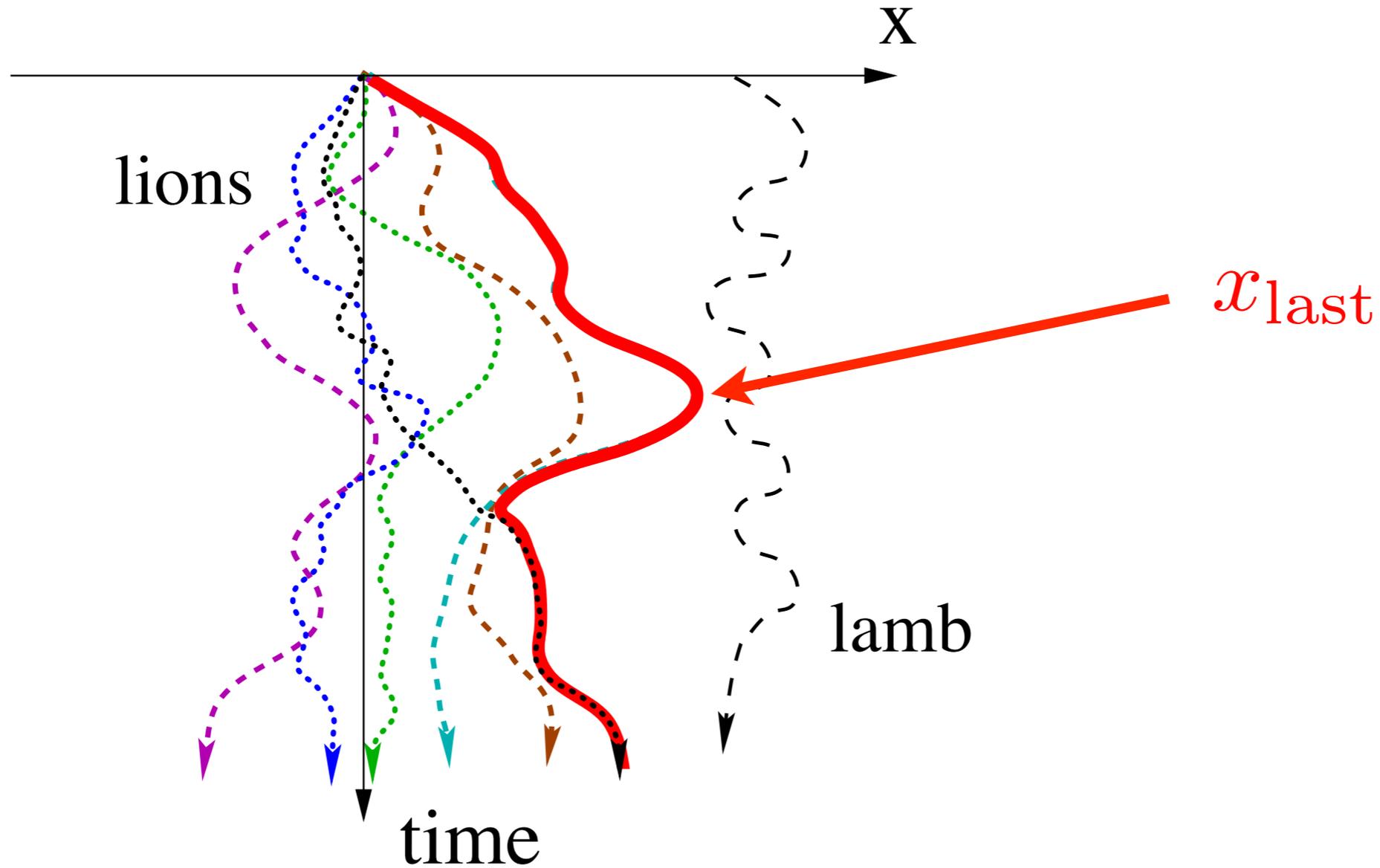
Many Diffusing Hunters, One Diffusing Prey

Krapivsky & SR (96, 99)



Many Diffusing Hunters, One Diffusing Prey

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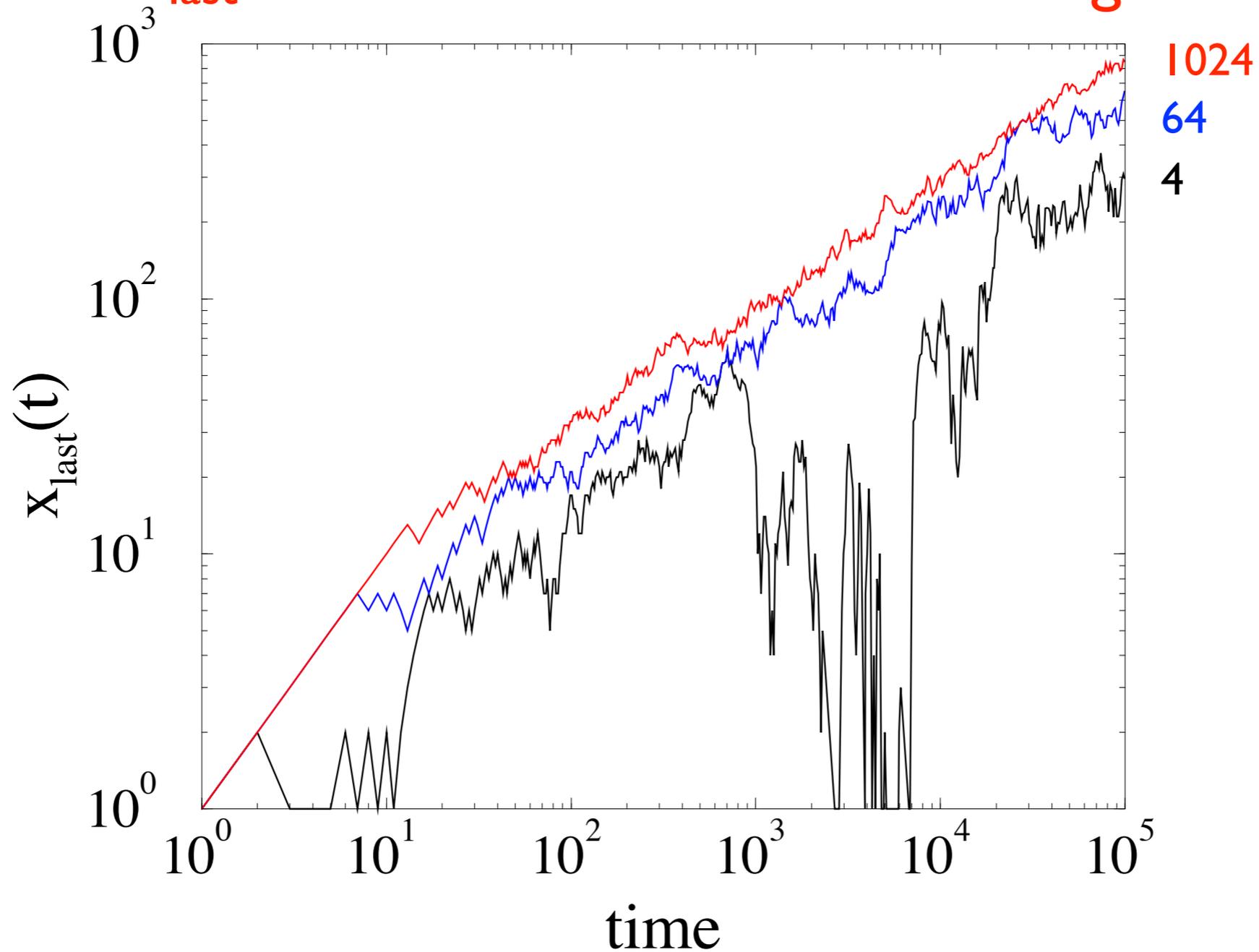


Many Diffusing Hunters, One Diffusing Prey

Krapivsky & SR (96, 99)



x_{last} becomes deterministic for large N



Many Diffusing Hunters, One Diffusing Prey

Krapivsky & SR (96, 99)



extremal criterion for x_{last} :

$$\int_{x_{\text{last}}}^{\infty} \frac{N}{\sqrt{4\pi Dt}} e^{-x^2/4Dt} dx = 1$$

$$x_{\text{last}}(t) \sim \sqrt{(4D \ln N) t} \equiv \sqrt{A_N t}$$

$N \gg 1$ lions

$$x_{\text{last}}(t) \sim \sqrt{2D \ln(c_0^2 Dt) t}$$

$N = \infty$ lions

constant density for $x < 0$
only $N \sim c_0 \sqrt{Dt}$ are dangerous

Many Diffusing Hunters, One Diffusing Prey

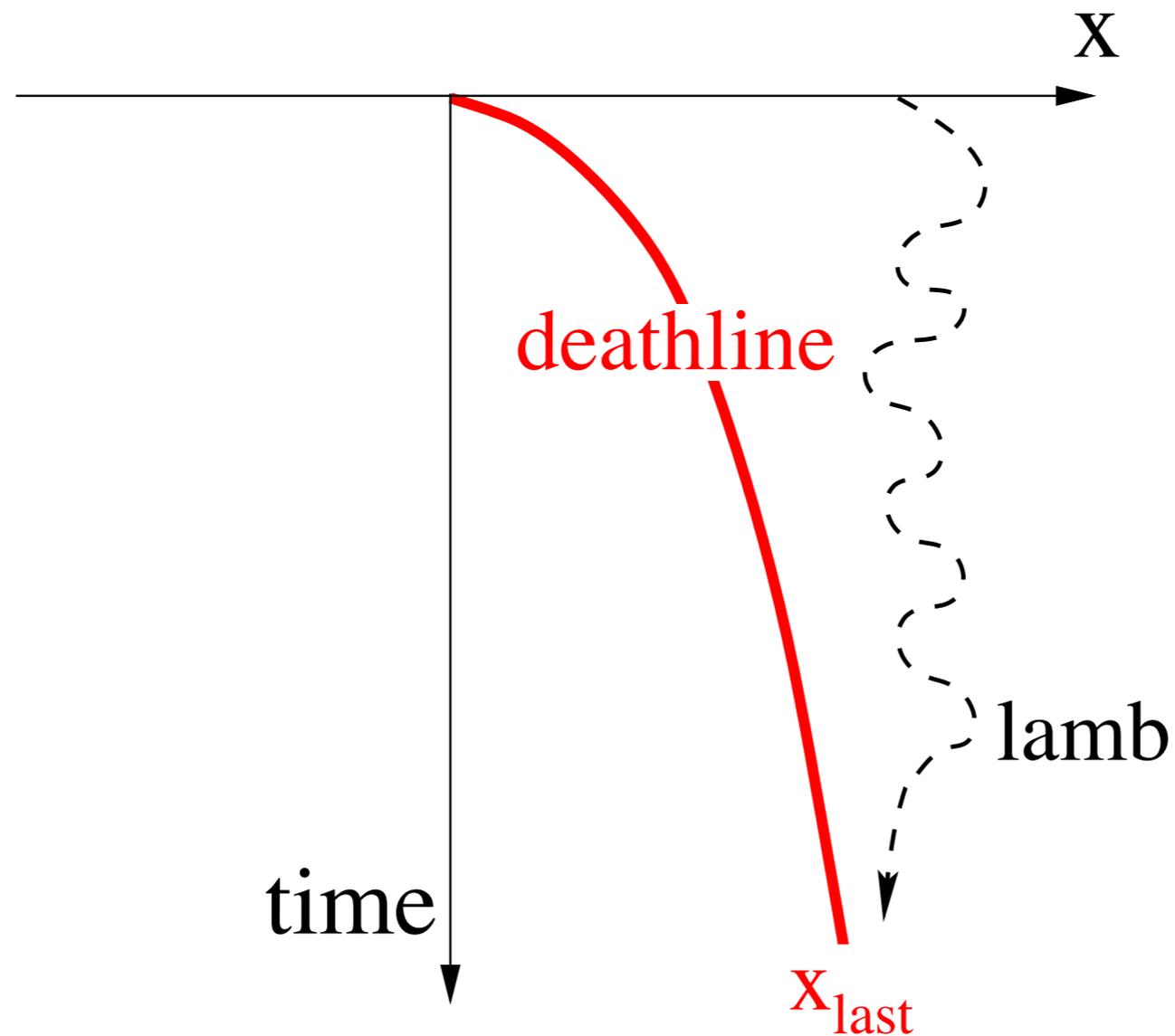
Krapivsky & SR (96, 99)



Effective Problem:
Deterministic Deathline

$$x_{\text{last}}(t) \sim \sqrt{(4D \ln N) t} \equiv \sqrt{A_N t}$$

$$x_{\text{last}}(t) \sim \sqrt{2D \ln(c_0^2 D t) t}$$



Many Diffusing Hunters, One Diffusing Prey

Krapivsky & SR (96, 99)



Effective Problem:
Deterministic Deathline

$$x_{\text{last}}(t) \sim \sqrt{(4D \ln N) t} \equiv \sqrt{A_N t}$$

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lamb probability distribution:

$$\frac{\partial p(x, t)}{\partial t} - \frac{x_{\text{last}}}{2t} \frac{\partial p(x, t)}{\partial x} = D \frac{\partial^2 p(x, t)}{\partial x^2} \quad (0 \leq x < \infty)$$

scaling ansatz: $p(x, t) \sim t^{-\beta_N - 1/2} F(\xi) \quad \xi = x/x_{\text{last}}$

$$\frac{D}{A_N} \frac{d^2 F}{d\xi^2} + \frac{\xi}{2} \frac{dF}{d\xi} + \left(\beta_N + \frac{1}{2}\right) F = 0$$

convert to Schrödinger equation:

define: $F(\xi) = e^{-\eta^2/4} \mathcal{D}(\eta)$ $\eta = \xi \sqrt{A_N/2D}$

$$\longrightarrow \frac{d^2 \mathcal{D}_{2\beta_N}}{d\eta^2} + \left[2\beta_N + \frac{1}{2} - \frac{1}{4}\eta^2 \right] \mathcal{D}_{2\beta_N} = 0$$

subject to $\mathcal{D}_{2\beta_N}(\eta) = 0$ $\eta = \sqrt{A_N/2D}$
 $\eta = \infty$

lowest energy eigenvalue
determined by the criterion:

$$2\beta_N + \frac{1}{2} \simeq \eta^2/4$$

$$\longrightarrow \beta_N \simeq A_N/16D$$

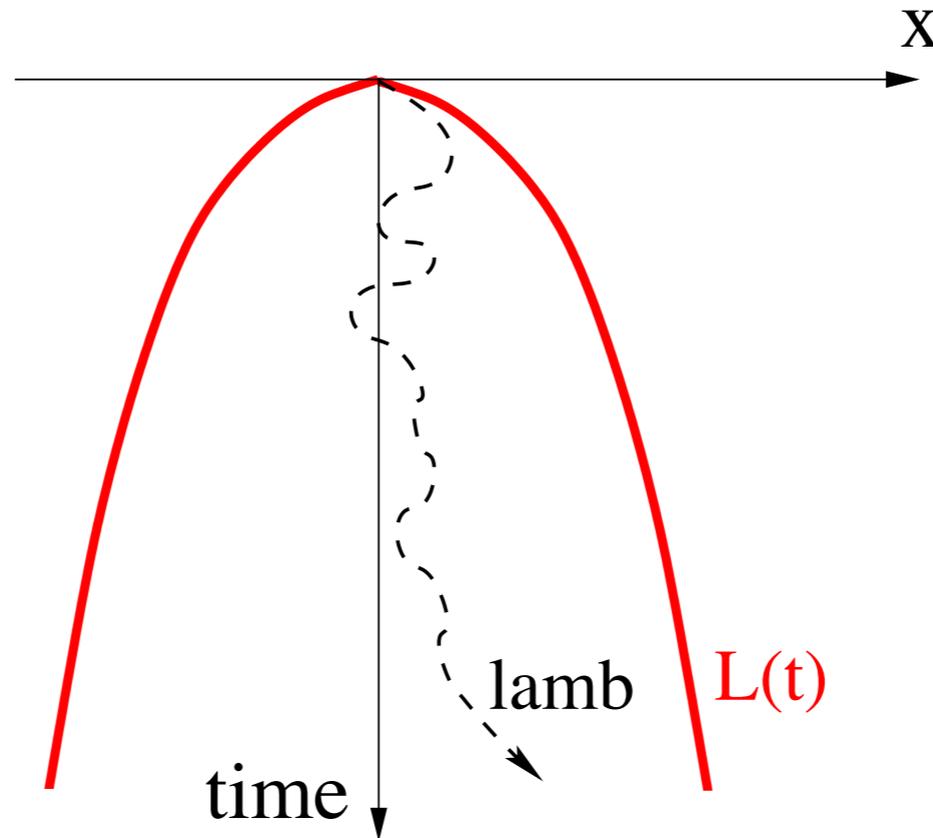
Summary

$$\beta_N \simeq \begin{cases} \frac{1}{4} \ln N & N \text{ finite} \\ \frac{1}{8} \ln t & N = \infty \end{cases} \quad S_\infty(t) \sim \exp\left(-\frac{1}{8} \ln^2 t\right)$$

Khintchine Iterated Logarithm Law

Khintchine (1924), Feller (1968), SR (01)

what is $L(t)$ so that the lamb “survives”?



if $L(t) > t^{1/2}$, $S_\infty > 0$ lamb survives

“free” approximation: $c(x, t) = \frac{S(t)}{\sqrt{4\pi Dt}} e^{-x^2/4Dt}$

$$\frac{\partial S}{\partial t} = -2D \frac{\partial c}{\partial x} \Big|_{x=L} = -\sqrt{\frac{L^2}{4\pi Dt}} e^{-L^2/4Dt} \frac{S}{t}$$

Khintchine Iterated Logarithm Law

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$$\text{if } L = \sqrt{At} \quad \ln S(t) = -\int_0^t \beta \frac{dt'}{t'} \quad \beta = \sqrt{\frac{A}{4\pi D}} e^{-A/4D}$$

if $A=\text{const.}$,
even if $A \gg 1$
 \rightarrow lamb dies

$$\text{if } L = \sqrt{4Dt f(t)} \quad \ln S(t) = -\int_0^t \sqrt{\frac{f(t')}{\pi}} e^{-f(t')} \frac{dt'}{t'}$$

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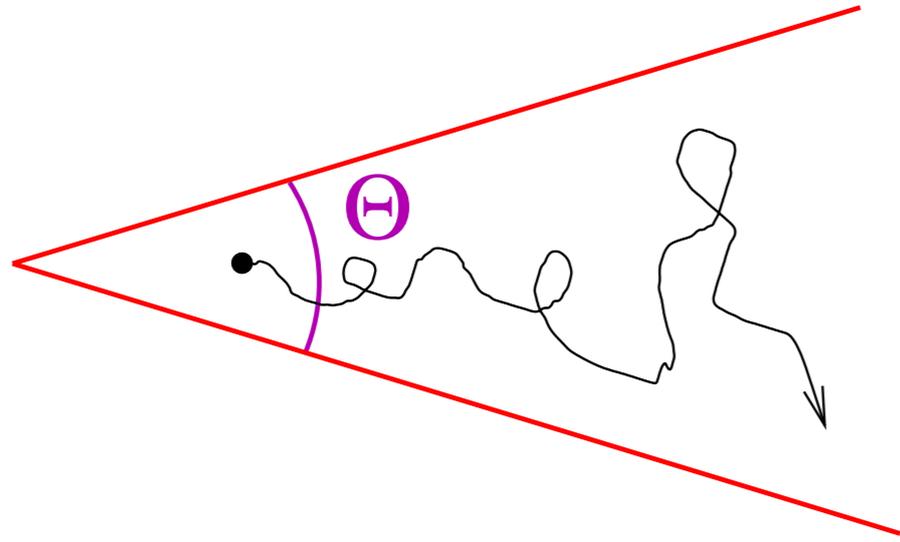
$$\begin{aligned} \text{if } L = \sqrt{4Dt f(t)} \quad \ln S(t) &= -\int_0^t \sqrt{\frac{f(t')}{\pi}} e^{-f(t')} \frac{dt'}{t'} \\ &= -\int_0^{\ln t} \sqrt{\frac{f(x)}{\pi}} e^{-f(x)} dx \end{aligned}$$

$$\text{for } f(x) = \lambda \ln x, \quad \rightarrow \quad \ln S \sim -\int^{\ln t} \frac{dx}{x^\lambda} \quad \text{marginal case } \lambda = 1$$

$$\text{when } L = \sqrt{4Dt \ln \ln t} \quad \rightarrow \quad S(t) \sim \frac{1}{\ln t}$$

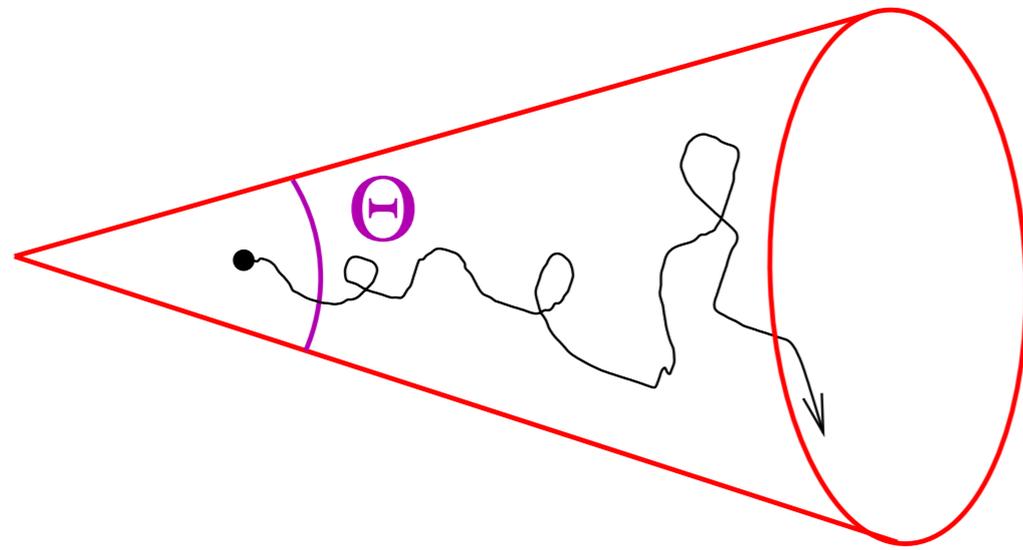
$$\text{ultimately } L(t) \sim \sqrt{4Dt(\ln \ln t + \frac{3}{2} \ln \ln \ln t + \dots)} \quad \rightarrow \quad S(t) \sim \frac{1}{\ln \ln \ln \dots \ln t}$$

Survival in Wedges



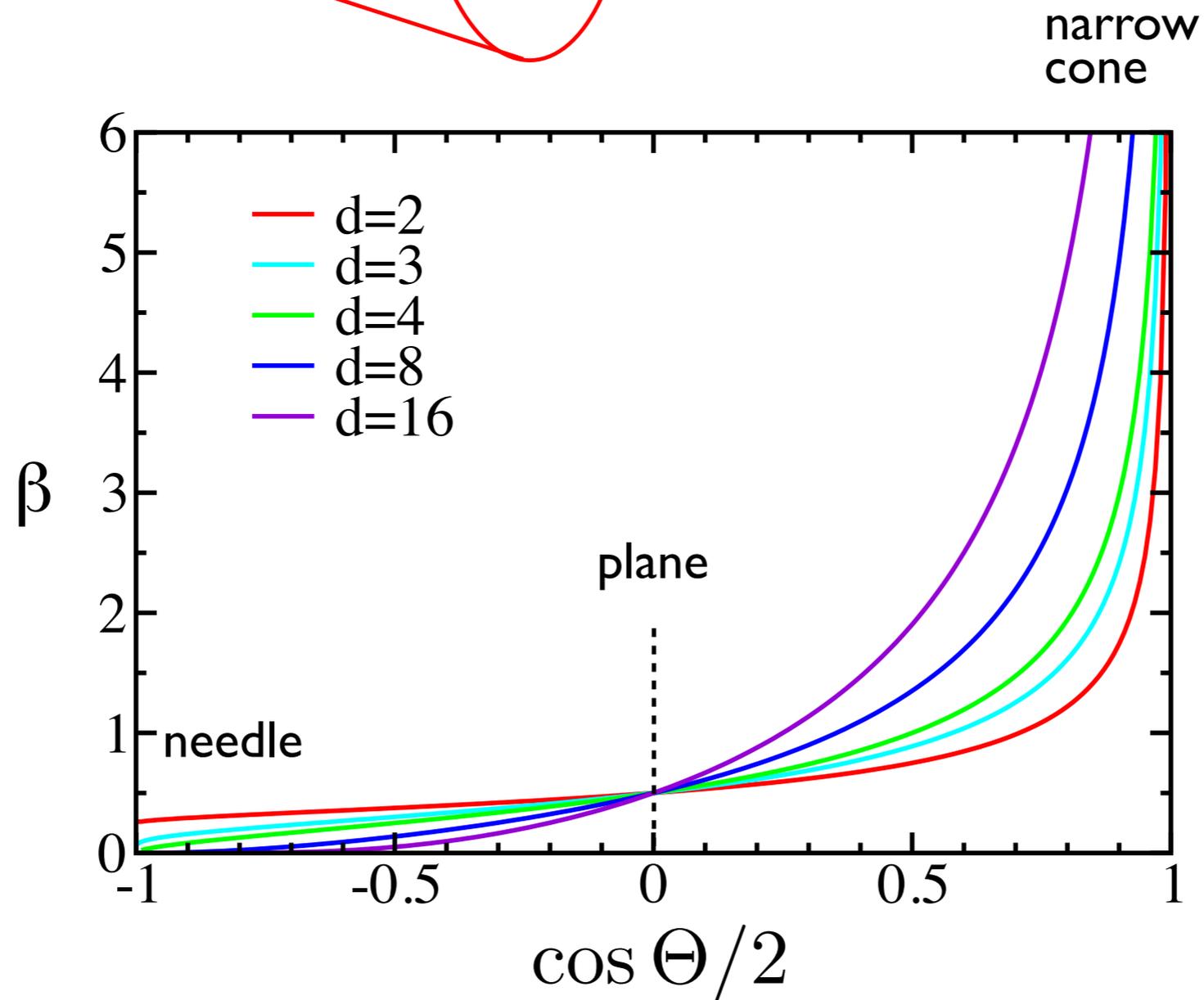
$$S(t) \sim t^{-\pi/2\Theta}$$

Survival in Wedges, Cones

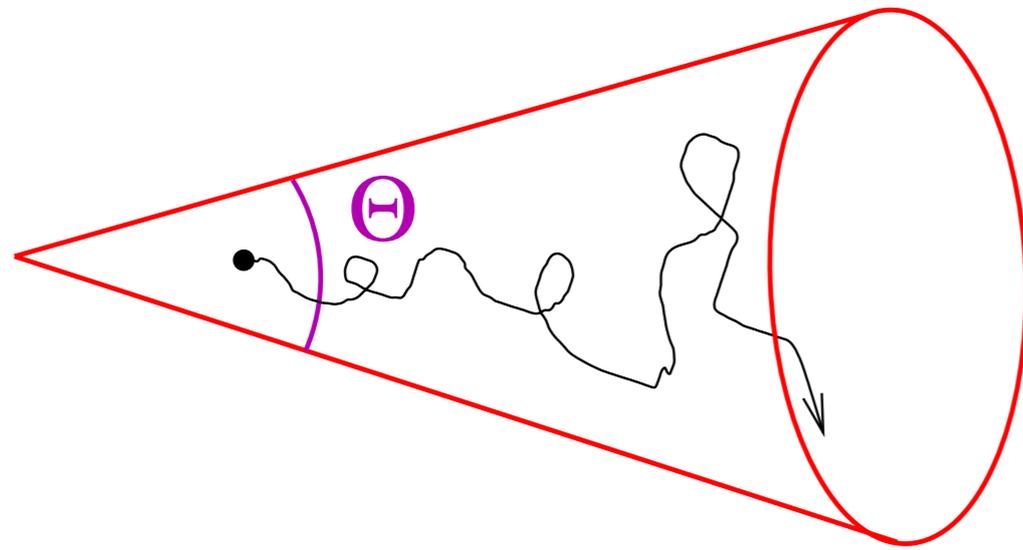


$$S(t) \sim t^{-\beta(\Theta, d)}$$

Ben-Naim and Krapivsky (2010)

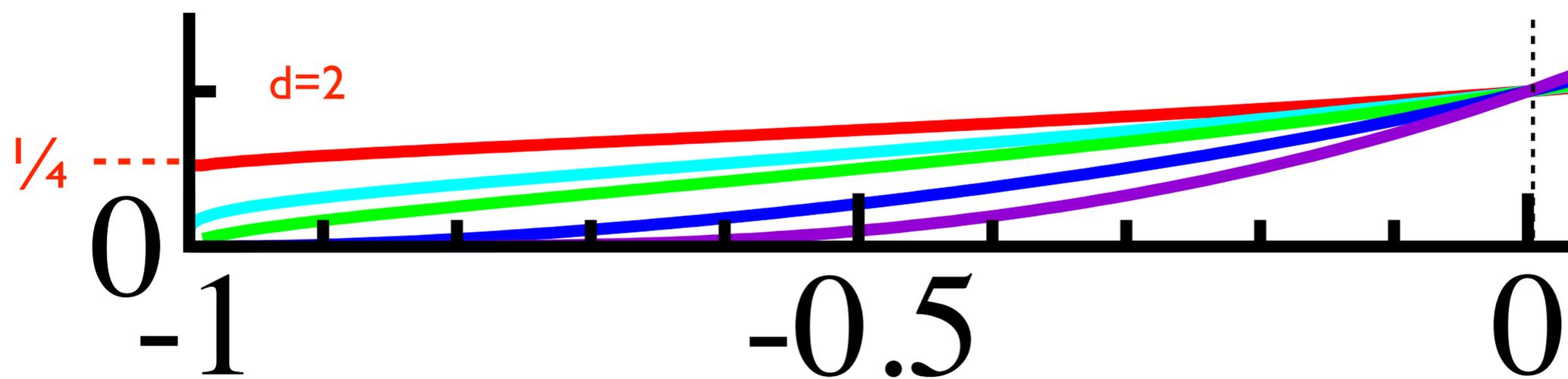


Survival in Wedges, Cones



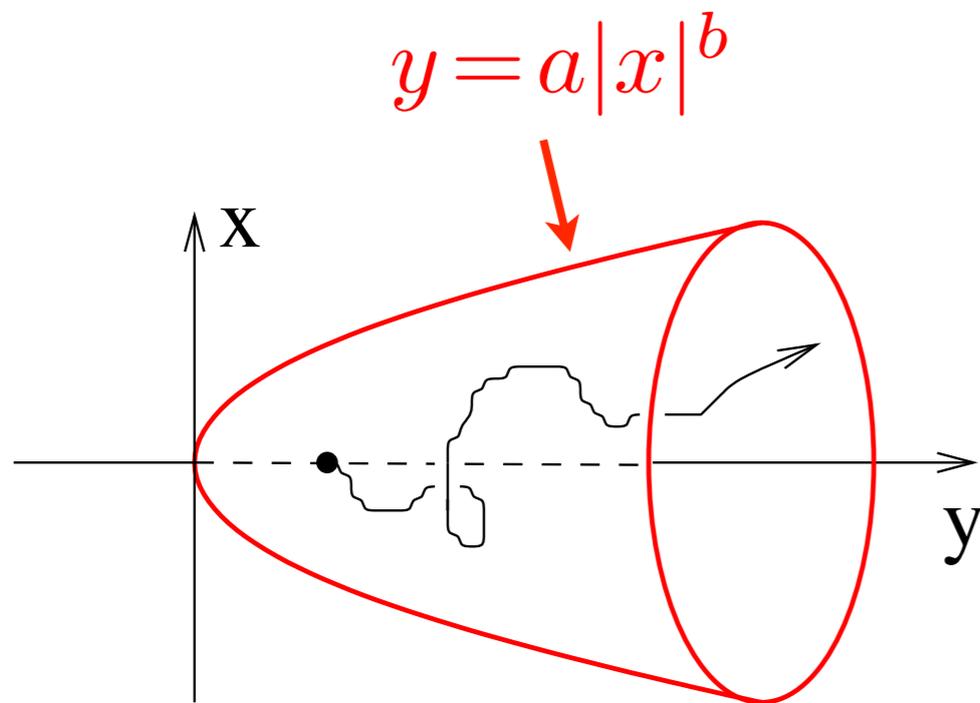
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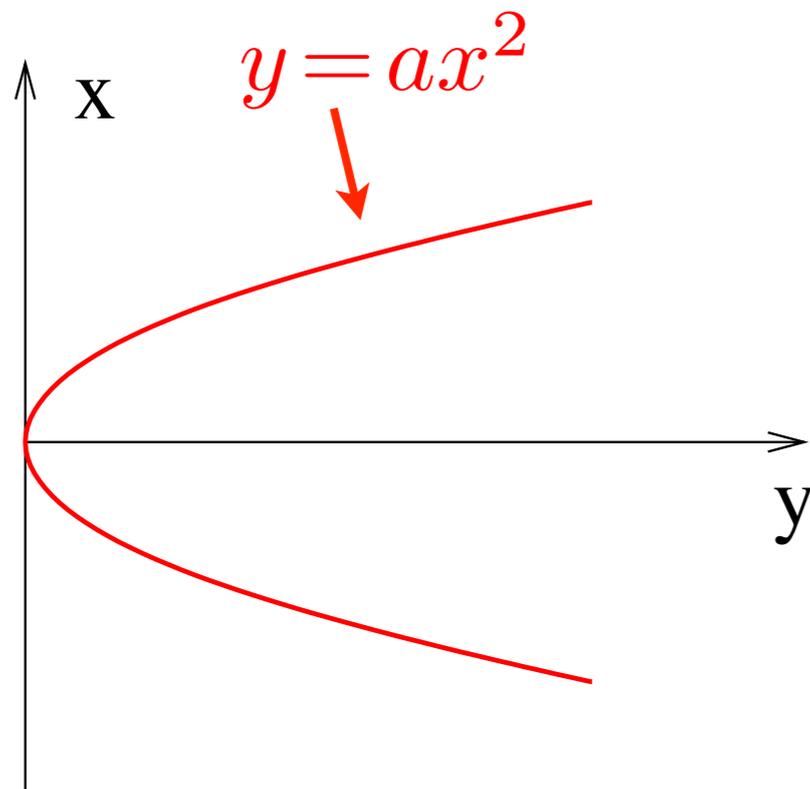
Survival in Wedges, Cones, and Paraboloids

Bañuelos et al (2001)
Lifshitz and Shi (2002)
KR (2010)



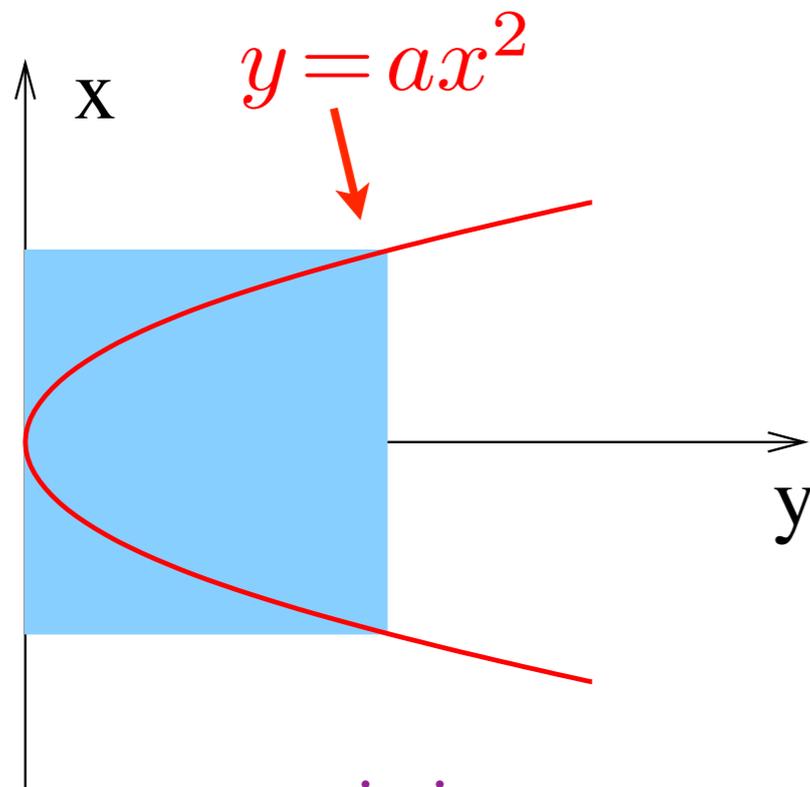
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Survival in Wedges, Cones, and Paraboloids

Bañuelos et al (2001)
Lifshitz and Shi (2002)
KR (2010)



survival criterion:

prob. to remain within $|x|$ and exit via right edge:

$$e^{-\pi^2 Dt / (2x)^2} \times e^{-y^2 / 4Dt}$$

optimize over y :

$$S(t) < \int_0^\infty e^{-\pi^2 Dt / (2x)^2} \times e^{-y^2 / 4Dt} dy$$

$$< \exp \left[-\frac{3}{4} \left(\frac{a\pi^2}{2} \right)^{2/3} (Dt)^{1/3} \right] \equiv \exp[-At^{1/3}]$$

$$A = \frac{3}{8} \pi^{4/3}$$

$$A_{\text{exact}} = \frac{3}{8} \pi^2$$

Survival in Wedges, Cones, and Paraboloids

generalized paraboloid in d dimensions:

$$y = a \left(\sqrt{x_1^2 + \dots + x_{d-1}^2} \right)^p \quad p > 1$$

the same approach as $d=2$ gives:

$$S < \exp \left[-\frac{p+1}{4} \left(\frac{4j_{(d-3)/2}^2 a^{2/p}}{p} \right)^{\frac{p}{p+1}} (Dt)^{\frac{p-1}{p+1}} \right]$$

Hitting Times in Wedges, Cones, and Paraboloids

backward equation: $T \equiv \langle t(\vec{r}) \rangle = \sum_{\vec{r}'} p_{\vec{r} \rightarrow \vec{r}'} [t(\vec{r}') + \delta t]$

continuum limit: $D\nabla^2 T = -1 \quad T|_{\text{boundary}} = 0$

relation to survival probability:
$$T = \int_0^\infty t F(t) dt = - \int_0^\infty t \frac{\partial S}{\partial t} dt$$
$$= \int_0^\infty S(t) dt$$

Hitting Times in Wedges, Cones, and Paraboloids

infinite wedge: if $S(t)$ decays faster than t^{-1} , then $T < \infty$

$$\Theta_c = \begin{cases} \pi/2 & d = 2 \\ 2 \cos^{-1}(1/\sqrt{3}) \approx 109.47 & d = 3 \end{cases}$$

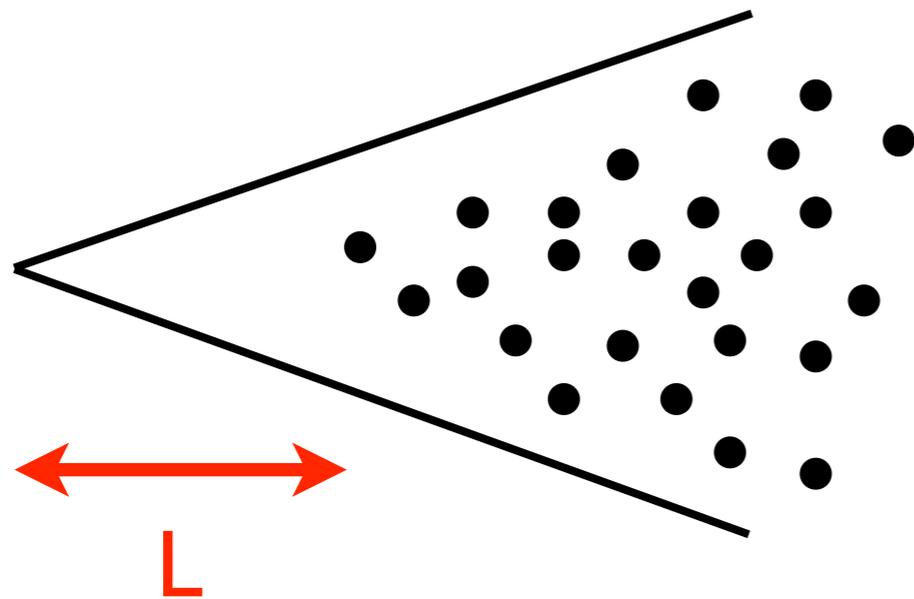


$$T(r, \theta) = \frac{r^2}{4D} \left(\frac{\cos 2\theta}{\cos \Theta} - 1 \right) + \sum_{n=0}^{\infty} A_n r^{\lambda_n} \cos(\lambda_n \theta) \quad \begin{aligned} \lambda_n &= (2n + 1) \frac{\pi}{\Theta} \\ A_n &= \frac{(-1)^{n+1} 4R^{2-\lambda_n}}{D\Theta\lambda_n(\lambda_n^2 - 4)} \end{aligned}$$

if $\Theta \geq \frac{\pi}{2}$, then $T \sim r^{\pi/\Theta} R^{2-\pi/\Theta}$ **divergent for $\Theta > \pi/2$**

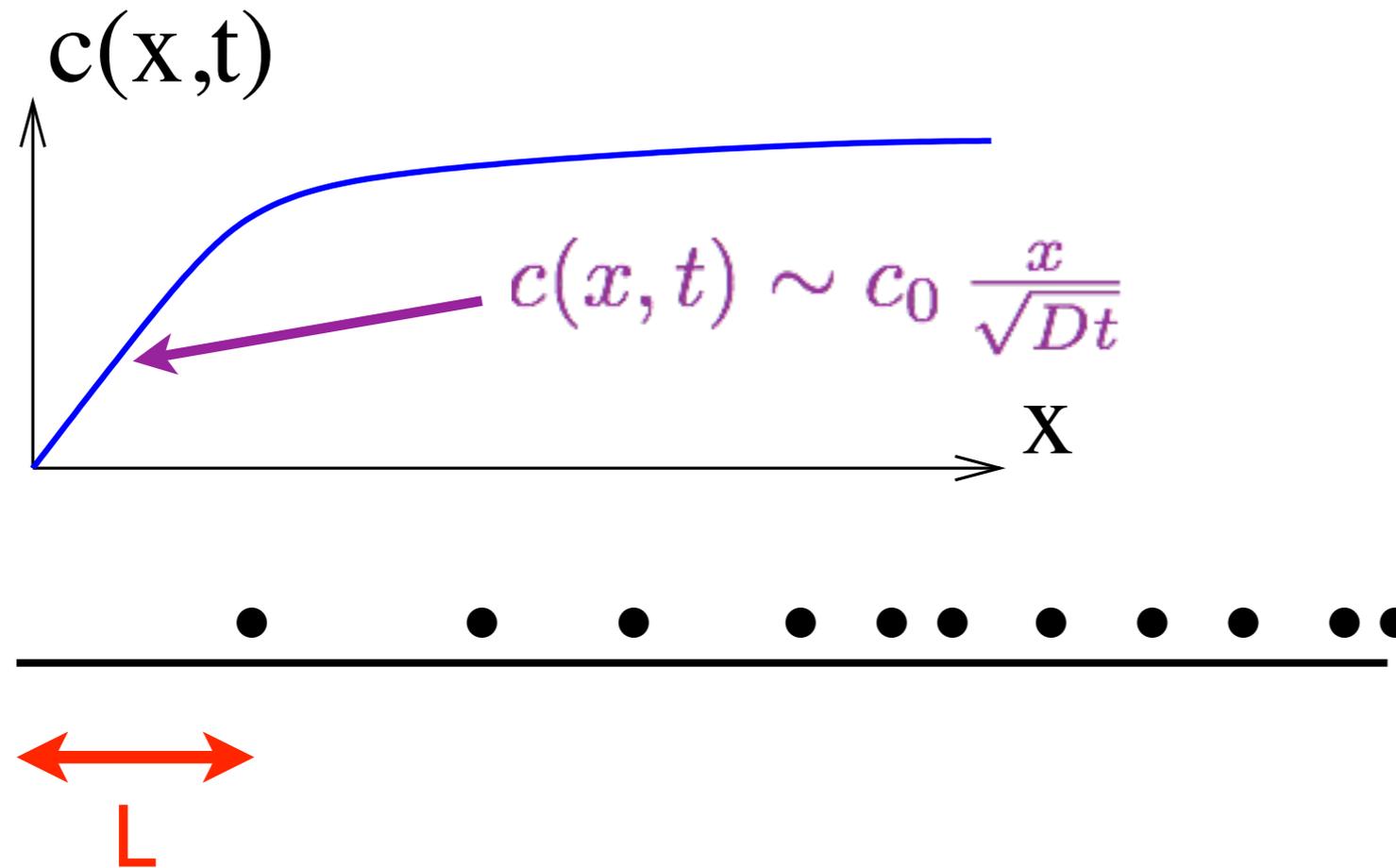
parabola: $T = \frac{1}{2D} (y - x^2)$

The Closest Particle



The Closest Particle

warm-up: 1d

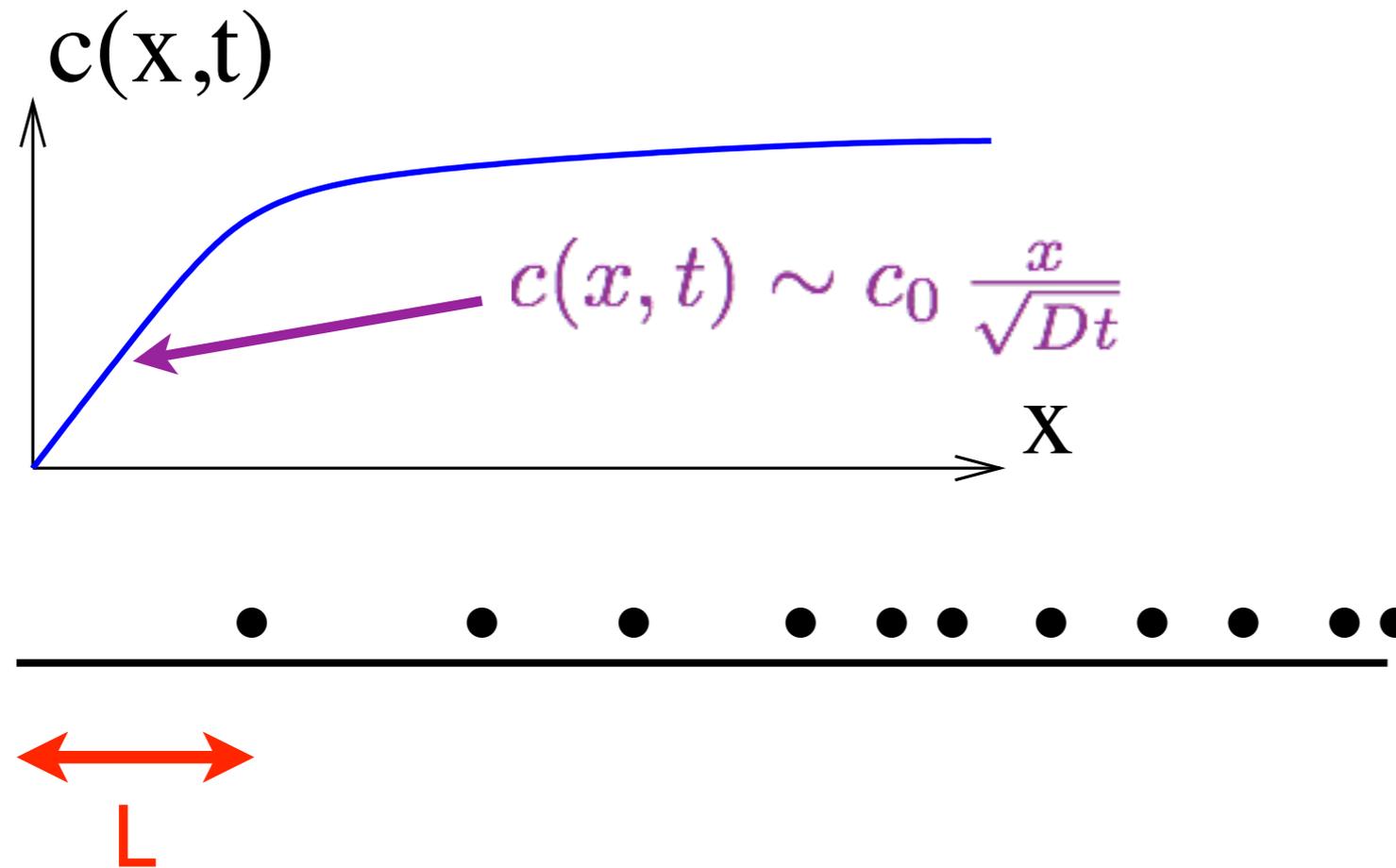


closest particle criterion:

$$\int_0^L c(x,t) dx = 1$$

The Closest Particle

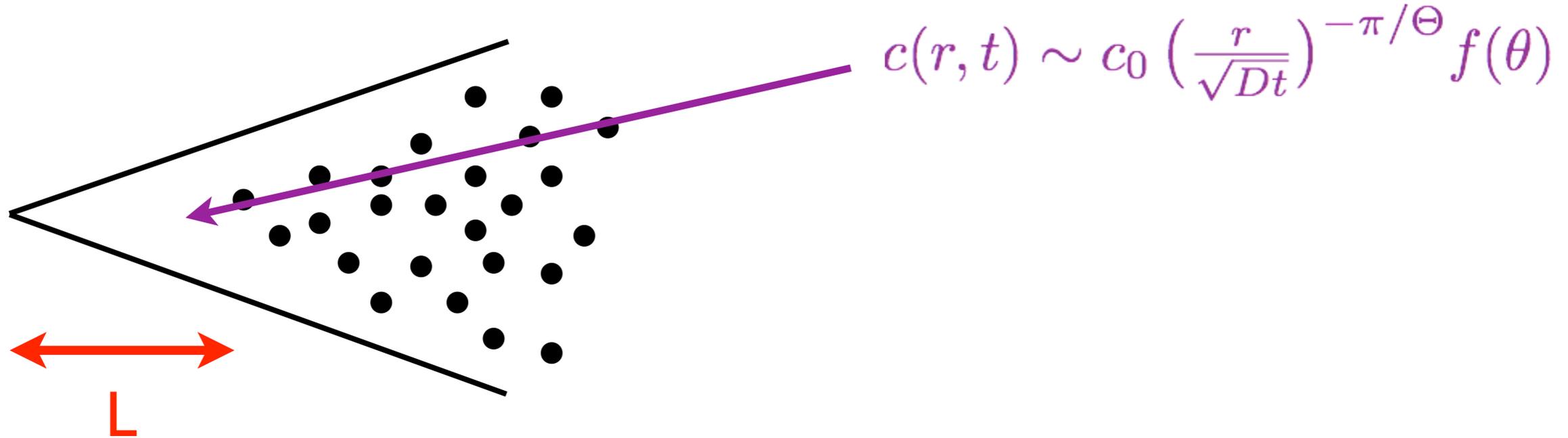
warm-up: 1d



closest particle criterion:

$$\int_0^L c(x,t) dx = 1 \rightarrow L \sim \left(\frac{Dt}{c_0^2}\right)^{1/4}$$

The Closest Particle



$$\int_0^L r c(r, t) dr \int_0^\Theta g(\theta) d\theta = 1 \rightarrow L \sim (Dt)^{\pi/(2\pi+4\Theta)} c_0^{-2\Theta/(\pi+2\Theta)}$$

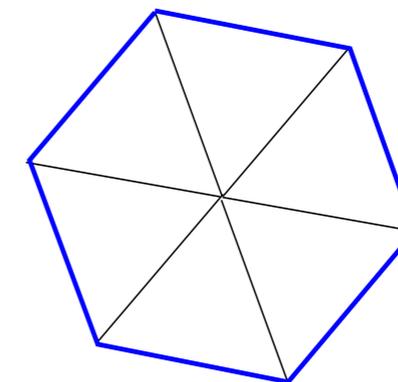
$$L \sim \begin{cases} (Dt)^{1/2} & \Theta \downarrow 0 & \text{crack} \\ (Dt)^{1/4} \rho^{-1/4} & \Theta = \pi/2 & \text{corner} \\ (Dt)^{1/6} \rho^{-1/3} & \Theta = \pi & \text{plane} \\ (Dt)^{1/10} \rho^{-2/5} & \Theta = 2\pi & \text{needle} \end{cases}$$

Some Concluding Remarks

- particle capture problems most interesting in 1d
- 3-particle problems well understood:

<i>no reversal, $123 \not\Rightarrow 321$</i>	$t^{-3/10}$
<i>3 never trails, $123 \not\Rightarrow 321, 312$</i>	$t^{-3/8}$
<i>3 always leads 1</i>	$t^{-3/6}$
<i>3 always leads 1 & 2</i>	$t^{-3/4}$
<i>order preserved</i>	$t^{-3/2}$

- several-particle problems partially understood:



Weyl chamber

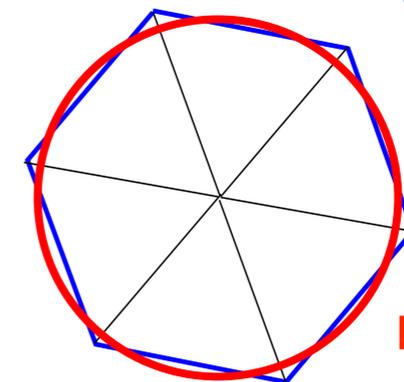
Some Concluding Remarks

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- several-particle problems partially understood:

*accurate but uncontrolled approximation
for survival probability*



Weyl chamber

- $N \rightarrow \infty$ approximation simple, powerful;
cheap “derivation” of iterated logarithm law

replace by
equal-area cone

Ben-Naim &
Krapivsky (2010)

Crass Commercialism

Aimed at graduate students, this book explores some of the core phenomena in non-equilibrium statistical physics. It focuses on the development and application of theoretical methods to help students develop their problem-solving skills.

The book begins with microscopic transport processes: diffusion, collision-driven phenomena, and exclusion. It then presents the kinetics of aggregation, fragmentation, and adsorption, where basic phenomenology and solution techniques are emphasized. The following chapters cover kinetic spin systems, by developing both a discrete and a continuum formulation, the role of disorder in non-equilibrium processes, and hysteresis from the non-equilibrium perspective. The concluding chapters address population dynamics, chemical reactions, and a kinetic perspective on complex networks. The book contains more than 200 exercises to test students' understanding of the subject. A link to a website hosted by the authors, containing an up-to-date list of errata and solutions to some of the exercises, can be found at www.cambridge.org/9780521851039.

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Eli Ben-Naim is a member of the Theoretical Division and an affiliate of the Center for Nonlinear Studies at Los Alamos National Laboratory. He conducts research in statistical, nonlinear, and soft condensed-matter physics, including the collective dynamics of interacting particle and granular systems.

Cover illustration: Snapshot of a collision cascade in a perfectly elastic, initially stationary hard-sphere gas in two dimensions due to a single incident particle. Shown are the cloud of moving particles (red) and the stationary particles (blue) that have not yet experienced any collisions. Figure courtesy of Tibor Antal.

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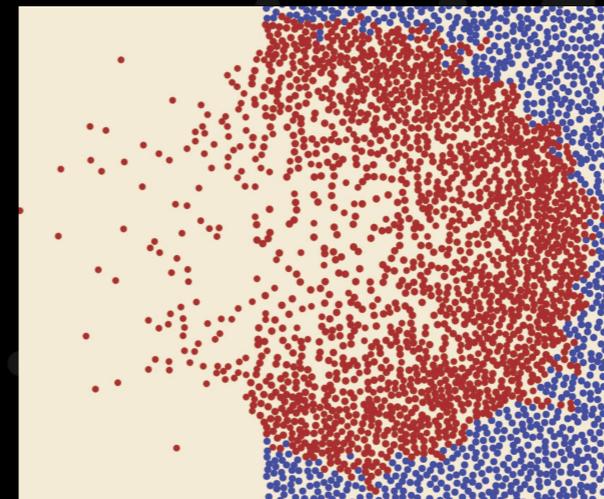


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Krapivsky
Redner
Ben-Naim

A Kinetic View of STATISTICAL PHYSICS

A Kinetic View of STATISTICAL PHYSICS



Pavel L. Krapivsky
Sidney Redner
Eli Ben-Naim

published
Dec. 2010

1. Aperitifs
2. Diffusion
3. Collisions
4. Exclusion
5. Aggregation

6. Fragmentation
7. Adsorption
8. Spin Dynamics
9. Coarsening
10. Disorder

11. Hysteresis
12. Population Dynamics
13. Diffusion Reactions
14. Complex Networks

> 200 problems