Turbulent Liquid Crystals KPZ Universality and the Asymmetric Simple Exclusion Process

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EXTREMES AND RECORDS
IPhT Saclay
June 14–17, 2011

Outline

- Stochastic growth processes

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- Exact solution of KPZ equation: Work of Amir-Corwin-Quastel & Sasamoto-Spohn
- Exact distribution from ASEP needed for KPZ analysis, C.T. & Widom (TW)

1.2 Measurement Techniques 5

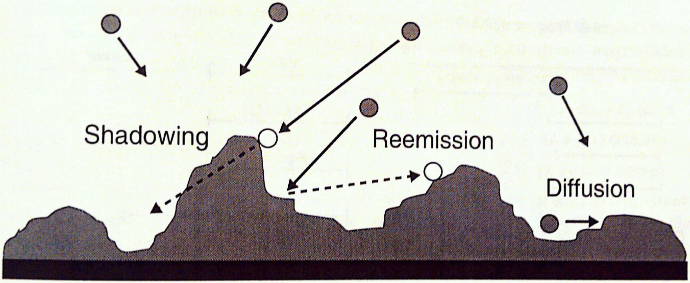


Fig. 1.3. Diagram of growth effects including diffusion, shadowing, and reemission that may affect surface morphology during thin film growth. The incident particle flux may arrive at the surface with a wide angular distribution depending on the deposition methods and parameters.

Figure: Want the (random) height function $h = h(x, t)$

Modelling Growth Processes

$$\frac{\partial h}{\partial t} = \Phi\left(h, x, t\right) + W(x, t)$$

$\Phi \longrightarrow$ captures growth effects to be modelled

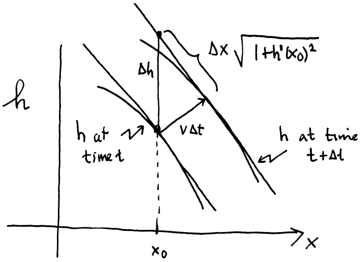
$W \longrightarrow$ noise term

This is a nonlinear stochastic PDE

Discrete versions are also popular models

Kardar-Parisi-Zhang–1986

Growth occurs normal to the surface



$$\begin{aligned} \Delta h &= h(x_0, t+\Delta t) - h(x_0, t) \\ &= v\Delta t \sqrt{1 + h'(x_0)^2} \\ &\approx v\Delta t \left[1 + \frac{1}{2} h'(x_0)^2\right] \end{aligned}$$

Thus want $\left(\frac{\partial h}{\partial x}\right)^2$ term in Φ.

KPZ Equation

$$\frac{\partial h}{\partial t} = \nu \underbrace{\frac{\partial^2 h}{\partial x^2}}_{\text{diffusion}} + \lambda \underbrace{\left(\frac{\partial h}{\partial x}\right)^2}_{\text{growth}} + \underbrace{W}_{\text{noise}}$$

- Nonlinear stochastic PDE.
- Difficult to make rigorous sense due to nonlinear growth term.
- KPZ made important prediction as $t \to \infty$

$$h(x,t) = \underbrace{v_\infty t}_{\text{deterministic linear growth}} + \underbrace{t^{1/3}}_{\frac{1}{3}\text{ fluctuations}} \chi$$

Famous KPZ $\frac{1}{3}$ exponent. χ is a fluctuating quantity—no prediction from KPZ phenomenology.

Experiments

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- An early experiment (2003 MYLLYS, TIMONEN,...) measured the "smouldering fronts in paper sheets" and determined that fluctuations were of order $1/3$ demonstrating growth is in KPZ universality class.

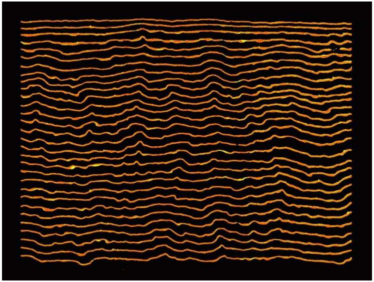

Figure: Digitized slow-combustion fronts with 10 s intervals. Courtesy of M. Myllys.

- TAKEUCHI & SANO, 2010: Convection of nematic liquid crystal driven by an electric field. They focus on the **interface between two turbulent states**. A thin square container is filled with a liquid crystal. The liquid crystal molecules, initially aligned perpendicular to the cell surfaces, strongly fluctuate when an AC voltage is applied leading to first turbulent state. A laser pulse nucleates a defect in the liquid crystal causing a second turbulent state.
- See K. A. Takeeuchi & M. Sano, *Universal Fluctuations of Growing Interfaces: Evidence in Turbulent Liquid Crystals*, PRL **104**, 230601 (2010), for experiments on droplet initial condition.
- To be published: K. A. Takeuchi & M. Sano: Same type of experiment but with flat initial condition. Please contact Dr. Takeuchi for details.

KPZ & Stochastic Heat Equation

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  1. Define solution to KPZ equation through a **Hopf-Cole transformation**

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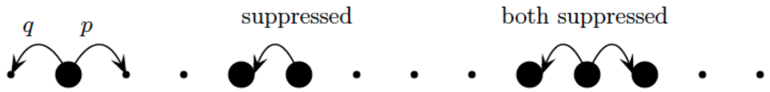
- For wedge initial conditions, in 2010 SASAMOTO/SPOHN and AMIR/CORWIN/QUASTEL carried this program out which required new theorems about the relation between stochastic heat equation and WASEP. Both groups used the ASEP results of TW which required a very delicate asymptotic analysis of the TW formula.

ASEP on Integer Lattice



- Each particle has an independent clock—when it rings with probability p (q) it makes a jump to the right (left) if site empty; otherwise, jump is suppressed.

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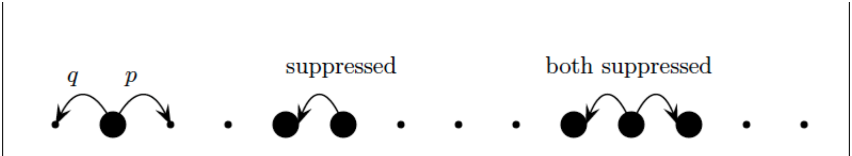

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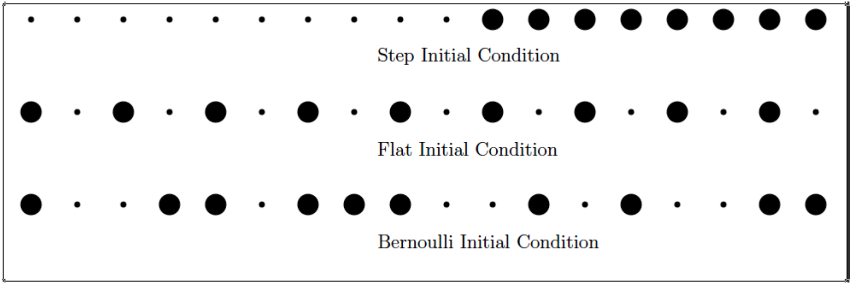
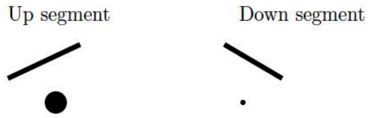
- Initial conditions:


Step Initial Condition

Flat Initial Condition

Bernoulli Initial Condition

Mapping to Growing Interface



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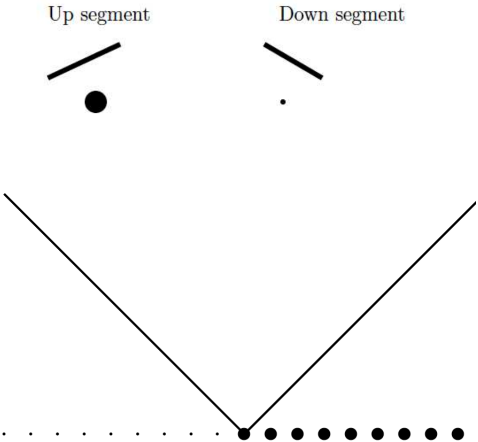

Initial height function corresponding to step initial condition

$$h(x, 0) = |x|$$

Discrete $Z_\varepsilon(T,X)$

Bertini & Giacomin, Sasamoto & Spohn, Amir, Corwin & Quastel:

$$Z_\varepsilon(T,X) = \frac{1}{2}\varepsilon^{-1/2}\exp\left[-\lambda_\varepsilon h(\frac{t}{\gamma},x) + \nu_\varepsilon \varepsilon^{-1/2} t\right]$$

where

$$t = \varepsilon^{-3/2}T,\ \ x = \varepsilon^{-1}X,\ \ \gamma = q - p = \varepsilon^{1/2}$$

$$\nu_\varepsilon = \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2,\ \ \lambda_\varepsilon = \varepsilon^{1/2} + \frac{1}{3}\varepsilon^{3/2}$$

Need ASEP formula for $h(t,x)$ and then let $\varepsilon \to 0$

$h(t, x)$ for ASEP
Step I—Transition probability for N-particle system

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$$\begin{aligned} \mathbb{P}_Y(X;t) &= \text{probability of configuration } X \text{ at time } t \\ &\quad \text{given the initial configuration is } Y \text{ at } t=0. \end{aligned}$$

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- Want solution to master equation that obeys the **initial condition**

$$\mathbb{P}_Y(X;0) = \delta_{X,Y}$$

Satisfying the initial condition is the hard part!

$\mathcal{S}_N$ denotes the permutation group on N symbols, $\sigma = (\sigma_1, \ldots, \sigma_N) \in \mathcal{S}_N$

Theorem (TW):

$$\mathbb{P}_Y(X;t) = \sum_{\sigma \in \mathcal{S}_N} \int_{\mathcal{C}} \cdots \int_{\mathcal{C}} A_\sigma(\xi) \prod_{i=1}^{N} \xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} e^{t\varepsilon(\xi_i)} \, d^N\xi$$

where

$$\begin{aligned}
\varepsilon(\xi_i) &= \frac{p}{\xi_i} + q\xi_i - 1 \\
A_\sigma(\xi) &= \prod_{\text{inversions } (\beta,\alpha) \text{ of } \sigma} S(\xi_\beta, \xi_\alpha) \\
S(\xi, \xi') &= -\frac{p + q\xi\xi' - \xi}{p + q\xi\xi' - \xi'} \\
\mathcal{C} &= \text{sufficiently small circle about zero} \\
& \quad \text{i.e. all poles of } A_\sigma \text{ lie outside of } \mathcal{C}
\end{aligned}$$

and each factor $d\xi_i$ carries a factor $\frac{1}{2\pi i}$.

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$$\sum_{\sigma} \mathrm{sgn}(\sigma) \left(\prod_{i<j} f(\xi_{\sigma(i)}, \xi_{\sigma(j)}) \times \frac{\xi_{\sigma(2)}\xi_{\sigma(3)}^2 \cdots \xi_{\sigma(N)}^{N-1}}{(1-\xi_{\sigma(1)}\xi_{\sigma(2)}\cdots\xi_{\sigma(N)})(1-\xi_{\sigma(2)}\cdots\xi_{\sigma(N)})\cdots(1-\xi_{\sigma(N)})} \right)$$

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where $f(\xi, \xi') = p + q\xi\xi' - \xi$

- Surprisingly this equals

$$p^{N(N-1)} \frac{\prod_{i<j}(\xi_j - \xi_i)}{\prod_i (1-\xi_i)}$$

Story behind proof of identity

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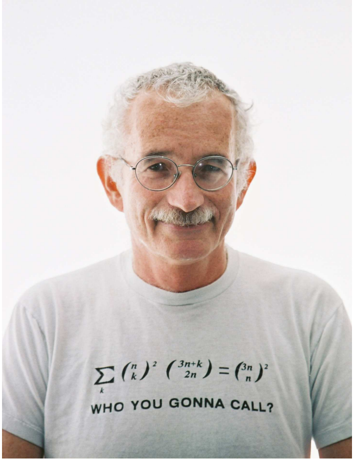
- 

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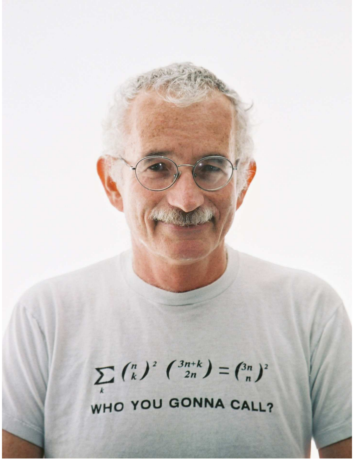
- 

- Doron saw the identity when it was still a conjecture and suggested to the authors that an identity of I. Schur (Problem VII.47 in Polya & Szegö) had a similar look about it and might be proved in a similar way. This led to the proof.

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$$\mathbb{P}(x_m(t) \leq x) = (-1)^m \sum_{k \geq m} \frac{1}{k!} \begin{bmatrix} k-1 \\ k-m \end{bmatrix}_\tau \tau^{m(m-1)/2 - mk + k/2} (pq)^{k^2/2}$$

$$\times \int_{\mathcal{C}_R} \cdots \int_{\mathcal{C}_R} \prod_{i \neq j} \frac{\xi_j - \xi_i}{f(\xi_i, \xi_j)} \prod_i \frac{\xi_i^x e^{t\varepsilon(\xi_i)}}{(1-\xi_i)(q\xi_i - p)} \, d\xi_1 \cdots d\xi_k$$

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  where $\left[{n \atop k}\right]_\tau$ is the τ-binomial coefficient and $\mathcal{C}_R$ is a large contour about zero, i.e. no poles outside of contour.
- Unfortunately, we are unable to perform an asymptotic analysis at this stage. Have similar formulas for other initial conditions.

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- With this determinant identity, recognize the kth term to be the kth term in the Fredholm expansion times some coefficients. This together with the τ-binomial theorem gives

$$\mathbb{P}_{\mathbb{Z}^+}(x_m(t) \leq x) = \int \frac{\det(I - \lambda K)}{\prod_{k=0}^{m-1}(1 - \lambda \tau^k)} \frac{d\lambda}{\lambda}$$

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- However, still unable to do asymptotic analysis! The operators K have exponentially large norms as $t \to \infty$.
- The idea is to replace K with operators with the same Fredholm determinant but better behaved norms.

Limit Theorems

Theorem (TW) Let $m = [\sigma t]$, $\gamma = q - p$ fixed, then

$$\lim_{t\to\infty} \mathbb{P}_{\mathbb{Z}^+}\left(x_m(t/\gamma) \le c_1(\sigma)t + c_2(\sigma)\, s\, t^{1/3}\right) = F_2(s)$$

uniformly for σ in compact subsets of $(0, 1)$ where $c_1(\sigma) = -1 + 2\sqrt{\sigma}$, $c_2(\sigma) = \sigma^{-1/6}(1 - \sqrt{\sigma})^{2/3}$.

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Theorem (ACQ, SS) Let

$$Z_\varepsilon(T,X) = p(T,X)e^{F_\varepsilon(T,X)}, \;\; p = \text{heat kernel}$$

then

$$F_T(s) = \lim_{\varepsilon\to 0} \mathbb{P}(F_\varepsilon(T,X) + \frac{T}{4!} \le s) = \text{KPZ crossover distribution}$$

Remark: Explicit formulas for $F_T(s)$.

Corollary(ACQ, SS)

$$\lim_{T\to\infty} F_T\left(2^{-1/3}T^{1/3}s\right) = F_2(s)$$

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Summary of KPZ Universality

- **Scaling exponent** $\frac{1}{3}$ does not depend upon initial configuration
- **Droplet initial conditions**: Long time one-point fluctuations described by $\mathbf{F_2}$.
- **Flat initial conditions**: Long time one-point fluctuations described by $\mathbf{F_1}$. Not (yet) a rigorous proof of this for KPZ equation.
- PRÄHOFER & SPOHN made these theoretical predictions concerning fluctuations on the basis of the PNG model.

Cast of Characters

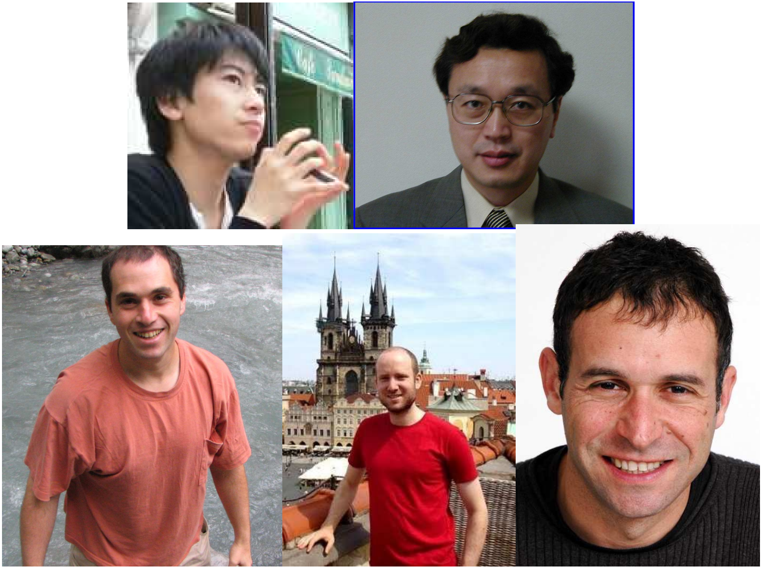

Figure: K. Takeuchi, M. Sano, G. Amir, I. Corwin, J. Quastel

Figure: T. Sasamoto, H. Spohn, H. Widom

Thank you for your attention