Universal behaviour of eigenvalues of the product of centered random Gaussian matrices near the edge

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Outline

- Part I
  - Introduction
  - Eigenvalue density of $X = X_1 X_2 \dots X_M$ (in the large N limit)
  - Surprising universality (for non-identical X_i's)
- Part II
  - Finite N corrections
  - Distribution of the largest module of the eigenvalue
- Summary

Non-hermitian Gaussian matrices

- Two i.i.d. GUE matrices: $A = A^\dagger$ and $B = B^\dagger$

$$d\mu(A,B) \propto DA\, DB\;\; e^{-\frac{N}{2\sigma^2}\mathrm{tr}A^2}\;\; e^{-\frac{N}{2\sigma^2}\mathrm{tr}B^2}$$

- Complex matrices

$$X = \frac{1}{\sqrt{2}}(A + iB) \quad , \qquad X^\dagger = \frac{1}{\sqrt{2}}(A - iB)$$

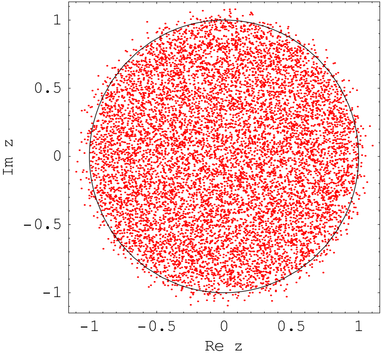
- Girko-Ginibre ensemble

$$d\mu(X, X^\dagger) \propto DX\; DX^\dagger\; e^{-\frac{N}{\sigma^2}\mathrm{Tr}XX^\dagger}$$

- Complex eigenvalues $z = x + iy$

$$\rho(x,y) = \begin{cases} \frac{1}{\pi\sigma^2} & \text{for } x^2 + y^2 \le \sigma^2 \\ 0 & \text{otherwise} \end{cases}$$

Illustration



- Monte-Carlo: 100 complex matrices 100-by-100
- points: eigenvalues
- solid: unit circle

Elliptic Gaussian measures

- Asymmetric mixing

$$X = \cos(\phi)A + i\sin(\phi)B\,, \quad X^\dagger = \cos(\phi)A - i\sin(\phi)B\,, \quad \tau = \cos(2\phi)$$

- Measure

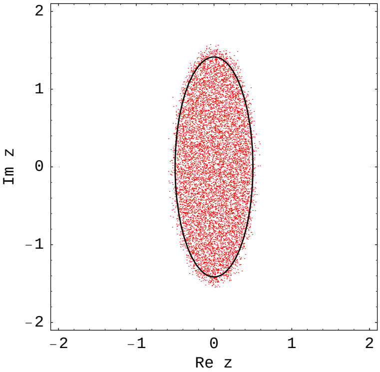
$$d\mu \propto DX\ DX^\dagger\ e^{-\frac{N}{\sigma^2(1-\tau^2)}\left(\mathrm{Tr}XX^\dagger - \frac{\tau}{2}\mathrm{Tr}\left(XX + X^\dagger X^\dagger\right)\right)}$$

- Crisanti, Sommers, Sompolinsky and Stein

$$\rho(x,y) = \begin{cases} \frac{1}{\pi\sigma^2(1-\tau^2)} & \text{for } \frac{x^2}{\sigma^2(1+\tau)^2} + \frac{y^2}{\sigma^2(1-\tau)^2} \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- The result holds also for real matrices

Illustration



- $\sigma = 1; \tau = -\frac{1}{2}$
- Monte-Carlo: 100 complex matrices 100-by-100
- $(x/a)^2 + (y/b)^2 = 1$; a=1/2; b=3/2;

Product of Gaussian matrices

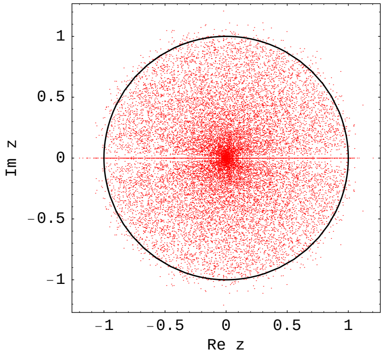
- Product of independent matrices $X = X_1 X_2 \dots X_M$
- Eigenvalue density

$$\rho(z) = \begin{cases} \frac{1}{M\pi\sigma^{\frac{2}{M}}} |z|^{-2+\frac{2}{M}} & \text{for } |z| \le \sigma \\ 0 & \text{for } |z| > \sigma \end{cases}$$

- Strong universality: X_i's do not have to be identical
- $\sigma = \sigma_1 \dots \sigma_M$; Result is independent of $\tau_1, \dots, \tau_M$!!!
- For $\sigma = 1$ and $M = 2, 3$

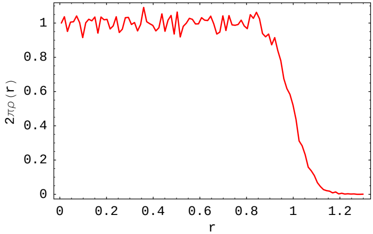
$$\rho_2(r) = \frac{1}{2\pi r} \,, \quad \rho_3(r) = \frac{1}{3\pi r^{4/3}}$$

Illustration



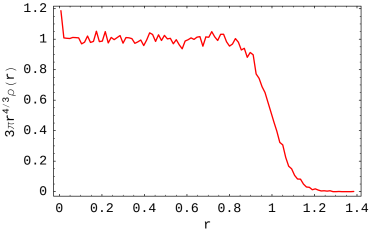
- Product of two GUE matrices $X = X_1 X_2$
- Monte-Carlo: 200 complex matrices 100-by-100
- accumulation of eigenvalues on the real axis, (Tr X is real); A. Edelman, J. Multivariate Anal. 60 (1997) 203.

Illustration



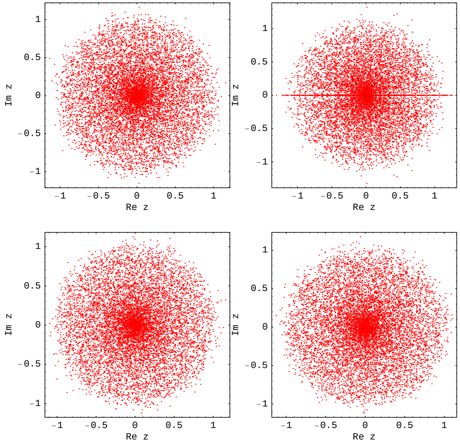
- Product of two GUE matrices $X = X_1 X_2$
- Radial profile $\rho_*(r) = 2\pi r \rho(r)$, where $r = |z|$
- Monte-Carlo: 1000 complex matrices 100-by-100

Illustration



- Product of two GUE matrices $X = X_1 X_2 X_3$
- Radial profile $3\pi r^{4/3}\rho(r)$, where $r = |z|$
- Monte-Carlo: 1000 complex matrices 100-by-100

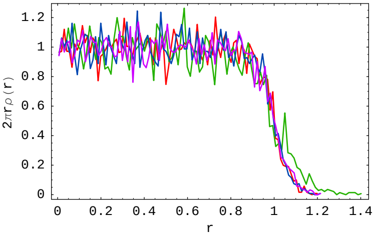
Illustration



- $X = X_1 X_2$
- MC 100, 100x100

1. G-G · G-G
2. RW · RW (unif. distr.)
3. GUE · G-G
4. GUE · Elliptic ($\tau = -1/2$)

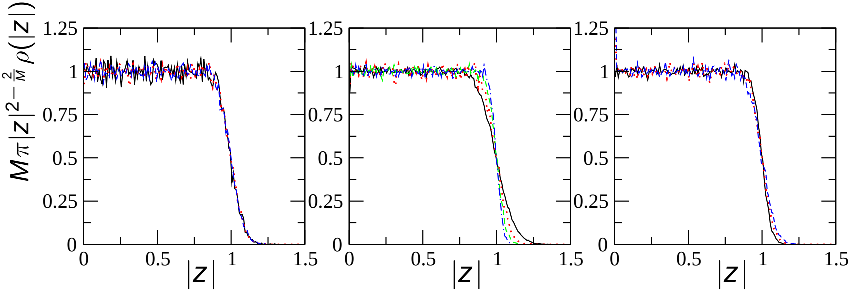
Illustration



- $X = X_1 X_2$; MC 1000 matrices 100x100

1. red: G-G $\cdot$ G-G
2. green: RW $\cdot$ RW (uniform distribution)
3. blue: GUE $\cdot$ G-G
4. violet: GUE $\cdot$ AC ($\tau = -1/2$)

Illustration



- left: $X = X_1X_2$ for GUE $\cdot$ GUE; G-G $\cdot$ G-G; Elliptic $\cdot$ GUE;
- middle: $X = X_1X_2$ for $N = 50, 100, 200, 400$;
- right: $X = X_1 \dots X_M$ for $M = 2, 3, 4$;

Universality

- Product of independent matrices $X = X_1 X_2 \ldots X_M$
- Eigenvalue density of X is rotationally symmetric even if densities of X_i's are elliptic !!
- Distribution is concentrated inside a circle $|z| \leq 1$

$$\rho(z) = \frac{1}{M\pi}|z|^{-2+\frac{2}{M}}$$

- Radial profile $r \in [0, 1]$

$$\rho_*(r) = 2\pi r \rho(r) = \frac{2}{M} r^{-1+\frac{2}{M}}$$

- The product of M iid G-G matrices has the same eigenvalue distribution as M-th power of one G-G matrix!

Linearization

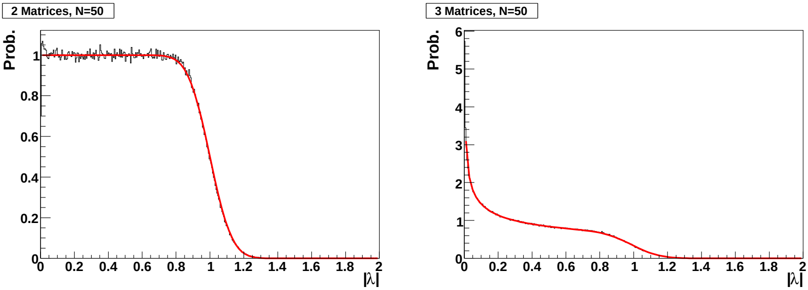
- $G(z) = \langle (z - X_1 X_2 \dots X_M)^{-1} \rangle \longleftrightarrow \mathcal{G}(w) = \langle (w - Y)^{-1} \rangle$;
$$Y = \begin{pmatrix} 0 & X_1 & & & 0 \\ 0 & 0 & X_2 & & 0 \\ & & \ddots & \ddots & \\ 0 & & & 0 & X_{M-1} \\ X_M & & & & 0 \end{pmatrix}$$
-
- For instance for $M = 3$
$$Y = \begin{pmatrix} 0 & X_1 & 0 \\ 0 & 0 & X_2 \\ X_3 & 0 & 0 \end{pmatrix} \rightarrow Y^3 = \begin{pmatrix} X_1X_2X_3 & 0 & 0 \\ 0 & X_2X_3X_1 & 0 \\ 0 & 0 & X_3X_1X_2 \end{pmatrix}$$
- Y^3 has the same eigenvalues as $X = X_1X_2X_3$, 3-fold degen.
- In general Y^M has the same eigenvalues as $X = X_1 \dots X_M$

Part II (Finite size effects)

Conjecture for finite N:

$$\rho_N(r) = \rho_*(r)\,\frac{1}{2}\mathrm{erfc}\left(a_N\,(r-1)\sqrt{2N}\right)$$

where $\rho_*(r) = \frac{2}{M} r^{-2+\frac{2}{M}}$, $\quad a_N \to a$



Why erfc?

- Jpdf

$$P(z_1, z_2, \ldots, z_N) = \frac{1}{\mathcal{Z}} \prod_{j=1}^{N} w(z_j) \prod_{i<j} |z_i - z_j|^2$$

- G-G : $w(z) = e^{-N|z|^2} \longrightarrow w(Z) = e^{-|Z|^2}$ where $Z = \sqrt{N}z$
- Pdf

$$\rho(Z) = \frac{1}{N\pi} e^{-|Z|^2} \sum_{n=0}^{N-1} \frac{|Z|^n}{n!} = \frac{1}{N\pi} \frac{\Gamma(N, |Z|^2)}{\Gamma(N)}$$

- Saddle point: $\rho(Z) = \frac{1}{2\pi N} \mathrm{erfc}\left(\sqrt{2}(|Z| - \sqrt{N})\right)$
- Rescaling: $\rho(z) = \frac{1}{2\pi} \mathrm{erfc}(\sqrt{2N}(|z| - 1))$

The largest absolute value of G-G

- Probability that all eigenvalues are in the circle of radius R:

$$E_N(R) = \frac{1}{\mathcal{Z}} \int_{|Z|<R} \prod_{j=1}^{N} d^2 Z_j \, w(Z_j) \prod_{i<j} |Z_i - Z_j|^2$$

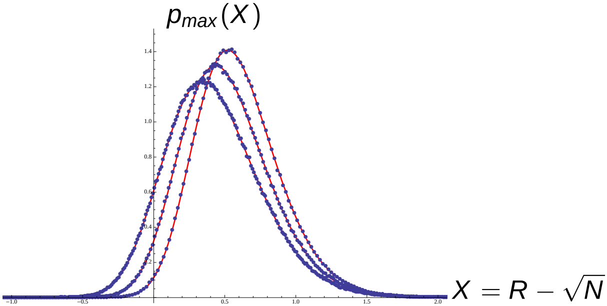
- Grobe, Haake, Sommers

$$E_{max}(R) = \prod_{k=1}^{N} \left(1 - \frac{\Gamma(k, R^2)}{\Gamma(k)}\right)$$

- Pdf for max: $p_{N,max}(R) = E'_N(R)$, $R = \sqrt{N} r$

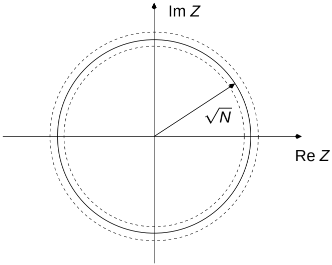
Monte Carlo vs Theory

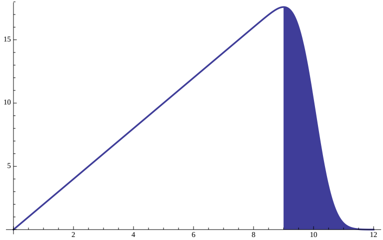
- $N = 25, 50, 100$
- $X = R - \sqrt{N}$



Approximation

$$\int d^2Z \, \frac{1}{2\pi}\frac{1}{2}\text{erfc}\left(\sqrt{2}(|Z| - \sqrt{N})\right) = \int dR \, R \, \frac{1}{2}\text{erfc}\left(\sqrt{2}(R - \sqrt{N})\right)$$

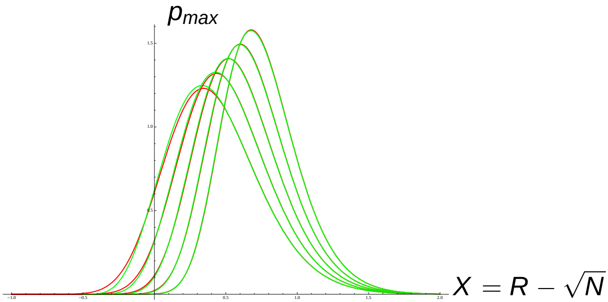




Number of eigenvalues in the strip: $n \sim \sqrt{N}$
$p_{max}(X) = (F_*^n(X))'$
$F_*(X)$ cdf in the strip

Monte Carlo vs Theory

- $N = 25, 50, 100, 200, 400$
- Exact (red) vs approximation (green)



Limiting distribution

- $X_N = R_N - \sqrt{N}$
- $F_*(X) \sim 1 - e^{-X^2} \left(\frac{1}{2X^2} - \frac{3}{4X^2} + \ldots \right)$
- Gumbel distribution

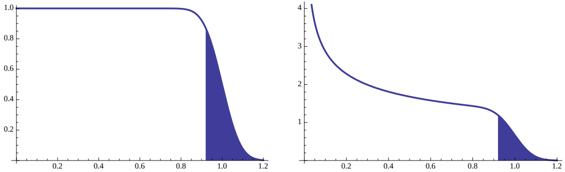
$$\Pr \left(\frac{X_N - B_N}{A_N} < \xi \right) \sim G(\xi)$$

- $A_N \sim \left(\frac{1}{2} \ln N \right)^{-0.5}$, $\quad B_N \sim \frac{1}{2} \ln N$
- Very slow approach to the limiting distribution
- Rescaling

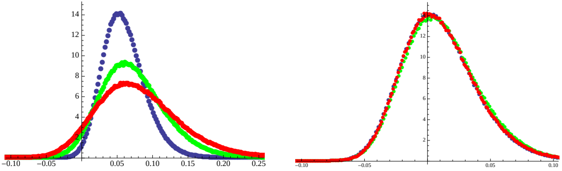
$$x_N = \frac{X_N}{\sqrt{N}} = r_N - 1, \; a_N = \frac{A_N}{\sqrt{N}}, \; b_N = \frac{B_N}{\sqrt{N}}$$

Largest eigenvalue distribution for the product

- $M = 2\ ,\ \ M = 3, \ldots$



- $p_{max,N}(x) = (F_*^n(x))', n \sim \sqrt{N}$
- Example: $M = 1, 2, 3, N = 100 \implies$ collapse



Summary

- The limiting distribution of $X = X_1 X_2 \ldots X_M$ for $N \to \infty$

$$\rho(z) = \begin{cases} \frac{1}{M\pi} |z|^{-2+\frac{2}{M}} & \text{for } |z| \leq 1 \\ 0 & \text{for } |z| > 1 \end{cases}$$

- Finite size corrections:
- Distribution of the absolute value of the largest eigenvalue: Gumbel
- Extensions:
  - Product of rectangular matrices
  - Free product of non-Hermitian matrices
  - Thank you!