Lecture 1. Superconducting Quantum Circuits

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Starting from the Josephson relation, we write the Lagrangian for a system of a Josephson junction shunted by a capacitor, i.e. a Transmon, and derive its Hamiltonian. Then, we study a system of capacitively or inductively coupled qubits, and discuss commonly used methods for solving their circuits quantum mechanically as well as approximated classically. In addition, we discuss a simple dynamical protocol that can be implemented to perform spectroscopy in the single-particle excitation sub-space.

1. SUPERCONDUCTIVITY AND JOSEPHSON RELATION

Giant atoms. One remarkable feature of superconducting qubits is that their energy-level spectra are governed by circuit element parameters and thus are configurable; they can be designed to exhibit "atom-like" energy spectra. Therefore, superconducting qubits are often referred to as artificial atoms, offering a rich parameter space of possible qubit properties, with predictable performance in terms of transition frequencies, anharmonicity, and complexity. The macroscopic size of these artificial atoms provides a major advantage as it allows lithographic wiring to be brought next to qubits to control their quantum degrees of freedom.

Josephson junctions are nonlinear, dissipationless circuit elements that form the backbone of superconducting circuits. A Josephson junction is formed by separating two superconducting electrodes with an insulator thin enough (a few nm for Al-Al oxide-Al junctions) such that Cooper pairs can tunnel through the barrier. The Josephson effect describes the super-current I that flows through the junction according to the equations

$$I = I_0 \sin(\varphi), \quad V = \frac{\Phi_0}{2\pi}\frac{d\varphi}{dt}, \tag{1}$$

where Φ_0 is the magnetic flux quantum ($= h/2e = 2 \times 10^{-15}$ Wb), I_0 is the critical current of the junction, and φ and V are respectively the superconducting phase difference ($= \varphi_1 - \varphi_2$) and voltage across the junction. The dynamical behaviour of these two equations can be understood by differentiating the current relation

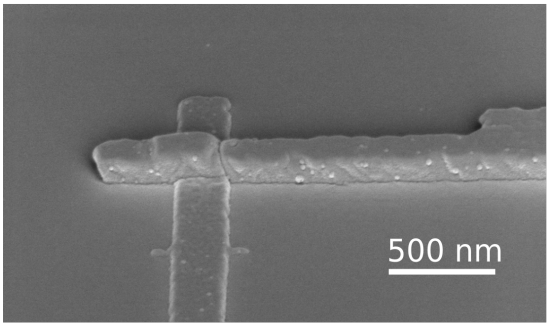
$$\frac{dI}{dt} = I_0 \frac{d\varphi}{dt}\cos(\varphi) \rightarrow \qquad \frac{dI}{dt} = I_0 \frac{2\pi}{\Phi_0} V \cos(\varphi) \rightarrow \qquad V = \frac{\Phi_0}{2\pi}\frac{1}{I_0\cos(\varphi)}\frac{dI}{dt} \tag{2}$$

By defining a Josephson inductance L_J according to the conventional definition of inductor ($V = L\, dI/dt$), one can see that Josephson junction is a non-linear inductor with inductance

$$L_J \equiv \frac{\Phi_0}{2\pi}\frac{1}{I_0\,\cos(\varphi)}. \tag{3}$$

Defining $E_J \equiv \Phi_0\, I_0/2\pi$, the energy stored in the junction is given by

$$\mathcal{U}_J = \int I\, V\, dt = \frac{\Phi_0}{2\pi}\int I\, d\varphi = -\frac{\Phi_0}{2\pi} I_0 \cos(\varphi) = -E_J \cos(\varphi). \tag{4}$$



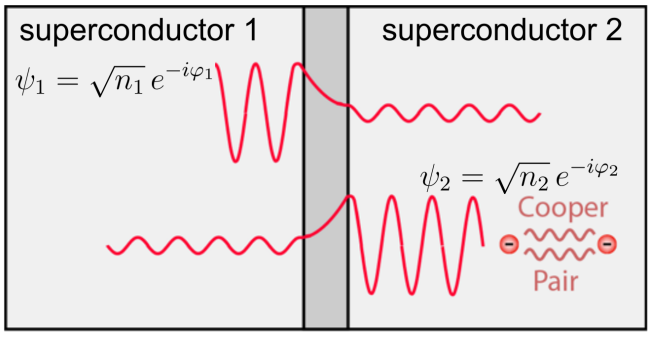


Figure 1. In a Josephson junctions the two superconducting islands are separated by a thin layer of insulator, (e.g. aluminum oxide, ∼5 nm). The Cooper pairs can coherently tunnel between the islands.

2. TRANSMON AS AN ANHARMONIC OSCILLATOR

LC oscillators. We begin with the classical description of a LC circuit. In this system, energy oscillates between electrical energy in the capacitor C and magnetic energy in the inductor L. It is customary to arbitrarily associate the electrical energy with the "kinetic energy" and the magnetic energy with the "potential energy" of the oscillator. To derive the classical Hamiltonian, we follow the Lagrange-Hamilton formulation. In this framework, one can represent the circuit elements in terms of one of its generalized circuit coordinates, charge or flux. In the following, we pick flux, defined as

$$\Phi(t) \equiv \int_{-\infty}^{t} V(t')\, dt'. \tag{5}$$

Given that we assume one end of the LC oscillator is kept at $V = 0$ (reference voltage, grounded, see Fig. 2), the voltage at the node is also the branch voltage across the element. The energy of the capacitor and inductor in terms of the node flux can be written as

$$\mathcal{T}_C = \frac{1}{2}CV^2 = \frac{1}{2}C\dot{\Phi}^2, \qquad \mathcal{U}_L = \frac{1}{2}LI^2 = \frac{\Phi^2}{2L}. \tag{6}$$

The Lagrangian of the system can be written

$$\mathcal{L}(\Phi, \dot{\Phi}) = \mathcal{T}_C - \mathcal{U}_L = \frac{1}{2}C\dot{\Phi}^2 - \frac{1}{2L}\Phi^2, \tag{7}$$

From the Lagrangian, one can derive the Hamiltonian using the Legendre transformation. We need to calculate the coordinate conjugate to the flux, which in this case is the charge on the capacitor.

$$Q = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C\dot{\Phi}, \tag{8}$$

The Hamiltonian of the system is

$$H(\Phi, Q) = Q\dot{\Phi} - \mathcal{L} = \frac{Q^2}{2C} + \frac{\Phi^2}{2L} = \mathcal{T}_C + \mathcal{U}_L, \tag{9}$$

as one would expect for a LC circuit. Note that the equations of motion gives what Kirchhoff rules provide

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\Phi}} - \frac{\partial \mathcal{L}}{\partial \Phi} = 0 \rightarrow \quad C\ddot{\Phi} + \Phi/L = 0 \quad \xrightarrow{C\ddot{\Phi} = C\dot{V} = \dot{Q} = I_C} \quad I_C + I_L = 0. \tag{10}$$

Here, currents are defined from high node to ground. One can check the correctness of Egn.(10) by noting that replacing $\Phi = Ae^{i\omega t}$ leads to the correct differential equation. The Hamiltonian described above is classical. In order to proceed to a quantum-mechanical description of the system, we need to introduce the quantum operators of charge and flux and impose their commutation relation:

$$[\hat{\Phi}, \hat{Q}] = i\hbar, \tag{11}$$

For convenience, we define the reduced flux $\hat{\varphi}$ and the reduced charge $\hat{\mathcal{N}}$ is the excess number of Cooper-pairs on the island and $[\hat{\varphi}, \hat{\mathcal{N}}] = i$. We can write the quantum Hamiltonian

$$\hat{H}_{LC} = 4\frac{e^2}{2C}\hat{\mathcal{N}}^2 + \left(\frac{\Phi_0}{2\pi}\right)^2 \frac{\hat{\varphi}}{2L}^2, \quad \text{LC oscillator,} \qquad \text{where} \quad \hat{\varphi} = 2\pi\hat{\Phi}/\Phi_0,\ \hat{\mathcal{N}} = \hat{Q}/2e \tag{12}$$

The charging energy $E_C = e^2/2C$ is the the required energy to add *each* electron of the Cooper-pair to the island, and $\Phi_0 = h/2e$ is the flux quantum.

Transmons. Capacitively-shunted Josephson junctions. The transmon circuit is obtained by replacing the linear inductor with a Josephson junction, which results in changing the potential term in the above Hamiltonian

$$\hat{H}_{\text{Qubit}} = 4\,E_C\,\hat{\mathcal{N}}^2 - E_J\,\cos(\hat{\varphi}), \quad \text{with} \quad E_C = \frac{e^2}{2C} \quad E_J = (\frac{\Phi_0}{2\pi})^2\frac{1}{L} = (\frac{\Phi_0}{2\pi})I_0 \tag{13}$$

For Transmons, $C = 80\,\text{fF}$ which give $E_C/h = 240\,\text{MHz}$, and $I_C = 40\,\text{nAmp}$ (SQUID inductance of 8 nH), which give E_J/h of 20GHz. These values suggests that considering Qubits as a linear oscillator and treating the non-linearity as a perturbation is a good approximation. In light of this, we define $\omega_p \equiv \sqrt{8\,E_J\,E_C}/\hbar$, and $\eta \equiv \sqrt{E_C/\,8E_J}$, with typical values of $\omega_p/2\pi$ =6 GHz and η =0.04,

$$\hat{H}_{\text{Qubit}}/\hbar = 4\eta\,\omega_p\hat{\mathcal{N}}^2 - \frac{\omega_p}{8\eta}\cos(\hat{\varphi}), \quad \text{Transmon qubit.} \quad \text{where} \quad \hbar\omega_p = \sqrt{8\,E_J\,E_C} \;\; \eta \equiv \sqrt{E_C/\,8E_J} \tag{14}$$

Recalling the quantum harmonic oscillator, $\hat{\mathcal{N}}$ and $\hat{\varphi}$ can be written in terms of lowering and raising operators

$$\hat{\varphi} = \sqrt[4]{\frac{8E_C}{E_J}}\,\frac{a+a^\dagger}{\sqrt{2}} = 2\sqrt{\eta}(a+a^\dagger), \qquad \hat{\mathcal{N}} = \sqrt[4]{\frac{E_J}{8E_C}}\,\frac{a-a^\dagger}{i\sqrt{2}} = \frac{a-a^\dagger}{4i\sqrt{\eta}}, \tag{15}$$

where a and $a^\dagger$ are lowering and raising operators in the space of eigenstates of quantum harmonic oscillator$|i\rangle$, with usual properties ($[a,a^\dagger]=1$, $a|m\rangle = \sqrt{m}\,|m-1\rangle$, $a^\dagger|i\rangle = \sqrt{m+1}\,|m+1\rangle$) and (truncated) matrix representation

$$a = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad a^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}. \tag{16}$$

Now that we derived the transmon Hamiltonian, we can see how the qubit non-linearity leads to the Hubbard U term in the Bose-Hubbard Hamiltonian. Given that $\eta \ll 1$,

$$\hat{H}_{\text{Qubit}}/\hbar\omega_p = 4\eta\left(\frac{a-a^\dagger}{4i\,\sqrt{\eta}}\right)^2 - \frac{1}{8\eta} + \frac{4\eta}{2\times 8\eta}(a+a^\dagger)^2 - \frac{16\eta^2}{24\times 8\eta}(a+a^\dagger)^4. \tag{17}$$

After dropping the constant term $1/8\eta$, noting that $aa^\dagger = a^\dagger a + 1$, and some algebra:

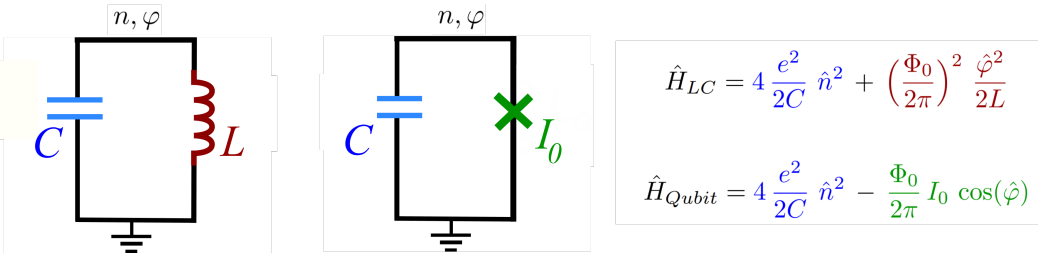


Figure 2. Schematic of a LC oscillator and a Transmon qubit. While LC Oscillator is linear (single frequency spacing), transmon is non-linear. This is achieved by replacing the linear inductor with a Josephson junction, a non-linear inductor.

$$\hat{H}_{\text{Qubit}}/\hbar\omega_p = a^\dagger\, a + 1/2 - \frac{\eta}{12}\,(a + a^\dagger)^4 \quad \text{(approximate, suitable for eigenvalues)} \tag{18}$$

To understand how non-linearity η appears as the Hubbard U, one needs to note that measurements are taking place in a rotating frame. Confining the discussion to the first three levels, a Single non-linear resonator Hamiltonian can be written as

$$\hat{H}_{\text{SQ}} = \omega_g|0\rangle\langle 1| + \omega_e|0\rangle\langle 1| + \omega_f|0\rangle\langle 1|. \tag{19}$$

Going to a rotating frame, R,

$$R = \exp(-i\,\omega_D\,t\begin{bmatrix}0&0&0\\0&1&0\\0&0&2\end{bmatrix}) = \begin{bmatrix}0&0&0\\0&\exp(-i\,\omega_D\,t)&0\\0&0&\exp(-2i\,\omega_D\,t)\end{bmatrix} \tag{20}$$

then the Hamiltonian in the rotating frame becomes

$$H_{Rot} = R^\dagger \hat{H}_{\text{SQ}}\, R + i\dot{R}^\dagger R = \begin{bmatrix}\omega_g&0&0\\0&\omega_e&0\\0&0&\omega_f\end{bmatrix} - \begin{bmatrix}0&0&0\\0&\omega_D&0\\0&0&2\omega_D\end{bmatrix} \tag{21}$$

Since any multiple of identity can be subtracted from Hamiltonian, we can recast H_{Rot} into more familiar forms,

$$H_{Rot} = \begin{bmatrix}0&0&0\\0&f_{10}-\omega_D&0\\0&0&f_{10}+f_{21}-2\omega_D\end{bmatrix} \tag{22}$$

As a result, in the absence of coupling and in the limit of considering only three levels per non-linear resonator (qutrit), the Hamiltonian takes the familiar form of the Bose-Hubbard becomes

$$H_{\text{Bose-Hubbard}} \rightarrow \sum \Delta_j\,\hat{n}_j - \frac{\eta\,\omega_p}{2}\sum(\hat{n}_j^2 - \hat{n}_j), \tag{23}$$

where $\Delta_j = f_{10} - \omega_D$, and the equation shows that non-linearity gives the Hubbard-U, $U_{\text{Hubbard}} = \eta\,\omega_p$.

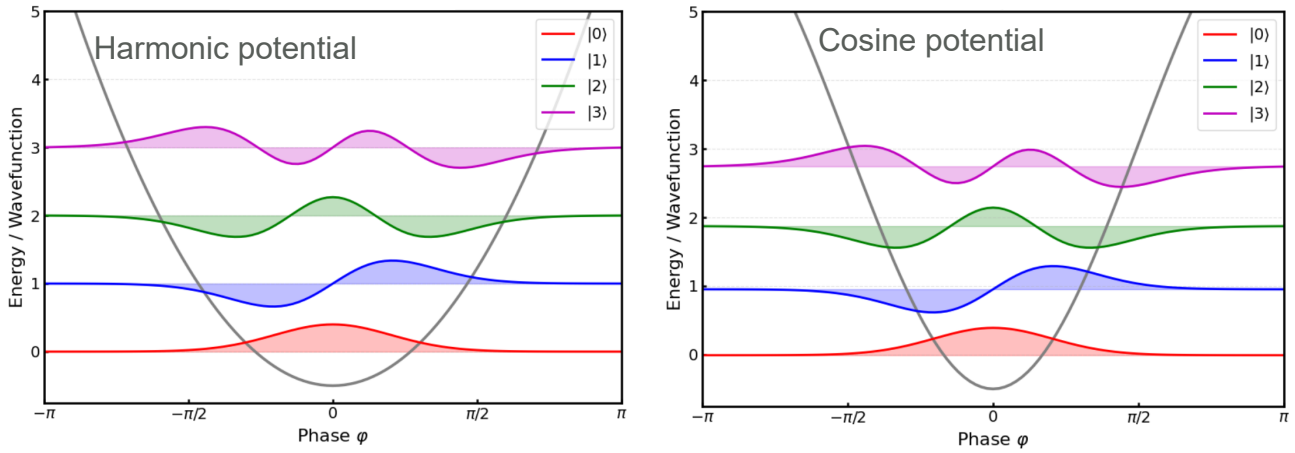


Figure 3. In an LC oscillator, the capacitance plays a role similar to mass in the mass-spring oscillator; as capacitance becomes larger, more levels fit inside the well. Note that potential is periodic in phase and so are the eigenstates. However, given that $E_C/E_J \ll 1$, the overlap between wells for $|0\rangle$, $|1\rangle$, $|2\rangle$ are exponentially small considering behavior in a single well is sufficient.

3. WHAT OSCILLATES ? THE PHYSICAL PICTURE AND PARAMETERS

Next, we discuss the physical meaning of the $|0\rangle$, $|1\rangle$, and $|2\rangle$ states of qubits. After proper understanding, one should have a physical picture for placing two photons into a non-linear resonator. When using $|0\rangle$, $|1\rangle$, and $|2\rangle$, are we referring to different modes or are we talking about number of excitations in a given mode ? the meaning of the $|0\rangle$ and whether one can detect any photon in this mode. Our non-linear resonators are conceptually similar to the ideal oscillators that Planck used in his model of the black-body's electromagnetic radiation. Following Planck's and Einstein's ideas, we could propose that our superconducting resonator is a quantum mechanical object that only exchanges discrete units of energy with the environment - our quanta or microwave photons. The states of the non-linear resonator would then be labeled by the number of photons that it stores , and the total energy would be a sum of quanta of energies.

The "photons" that we find in the superconductor have a mixed nature: They are excitations of the electromagnetic field that are accompanied by waves of charge and current on the surface of the metal. This hybrid nature would make it more appropriate to call them *microwave plasmons*. These microwave photons do have a matter-like component. This component allows us to control the properties of the photon - frequency, speed, wave-length, etc. - by a proper design of the circuit: capacitance, inductance, thickness of the material, etc. Moreover, the matter component in the wave enables the strong interaction between these microwave photons and other circuit excitations.

Distributed vs. lumped-element resonators. A mode is a spatiotemporal pattern of vibration, e.g., a standing wave in a string. Classical oscillators [1], such as a violin string can be in any of their oscillation modes, and the amplitude of oscillation of each mode (mode energy) is a continuous variable. In quantum systems, such as photons in cavities, the occupancy of each mode is quantized. The state of the system can be written with the occupancy of one of these modes. For example, $|1,\,5,\,0,\,2,\,0,\,0\rangle$ means there is 1 photon in the first mode, 5 photons in the second mode, no photon in the third mode, and so on. Therefore, the $|0\rangle$, $|1\rangle$ state of qubits, refer to number of occupancy of a given mode (usually lowest one) with photons ($|0,\,0,\,0,\,0,\,0,\,0\rangle$ and $|1,\,0,\,0,\,0,\,0,\,0\rangle$), and not occupying of different modes ($|0,\,0,\,0,\,0,\,0,\,0\rangle$ and $|0,\,1,\,0,\,0,\,0,\,0\rangle$). The lumped element assumption implies that the circuit only supports a fixed spatial distribution of charge and flux, it only has a single mode or frequency, which is why sometimes it is called a zero mode resonator.

A natural question to ask is if we can make a qubit by using a fundamental mode of an interrupted co-planar wave-guide (CPW) to form a LC oscillator ? (Fig. 4) In principle, one could choose $|0\rangle$ and $|1\rangle$ to be the zero or 1 photon in that mode, respectively. If we decide to place the fundamental frequency conveniently at 6GHz, the wavelength $\lambda = v/f$ becomes $\lambda = 3 \times 10^8 (m/s)/\sqrt{5.5}/6$ GHz=2 cm, and size of the resonator would be $\lambda/2 = 1$ cm. This a good size (and used for our readout resonators). However, this oscillator is not anharmonic and has uniform energy level spacing. This won't be a practical qubit, since cannot be driven properly.

In Transmon qubits, the resonance frequency of the CPW part alone is about 30 GHz (due to its short length). Therefore, at commonly chosen frequency of operation (6 GHz) the CPW part acts as a capacitor. Given that inductance of junctions are large and concentrated to a small point in space, with a good approximation, there is a spatial separation between the capacitor-like and inductor-like part of this oscillator. This is exactly how LC oscillators in an electrical engineering textbook are modeled: lumped-elements, where L and C are points without any spatial extent (similarly for mass-spring oscillators). The lumped element approximation means each component is treated as if all of its energy is stored in one place. Consequently, we say that the Josephson-based qubits are lumped-element resonators.

The lumped-element approximation brings design convenience, but does not have fundamental physics importance. In practice, this mode of oscillation also has some spatiotemporal pattern of oscillatory electromagnetic wave, which we choose to ignore. This is a good approximation; however the goodness of the lumped-element approximation is inconsequential. There is nothing in the physics that we want to exploit that demands this approximation to be accurate. The negative consequence of using the lumped-element phrase is to loose sight of the fact that every oscillator has spatial extend and hence it supports other modes of oscillation. The other modes of transmon are much higher in energy (the next mode is > 60 GHz, which is a $\lambda/2$ mode). Deviation from the lumped-element approximation, such as that the small capacitance of the junction or the inductance of the CPW part and the fact that it has spatial extend, leads to shift of physical parameters from design parameters.

If a resonator is modeled as a lumped-element one, it means that the resonator is built such that kinetic energy and potential energy are concentrated in spatially separate places. Usually one can see how this mode comes about by modifying parameters of the fundamental mode, as frequently there is an analytical continuation from fundamental mode to lump-element behavior. Given that higher modes usually involve relative motion of internal degree of freedom, then they become much higher in energy and hence inaccessible. This matter become more clear after solving **Problem 9**.

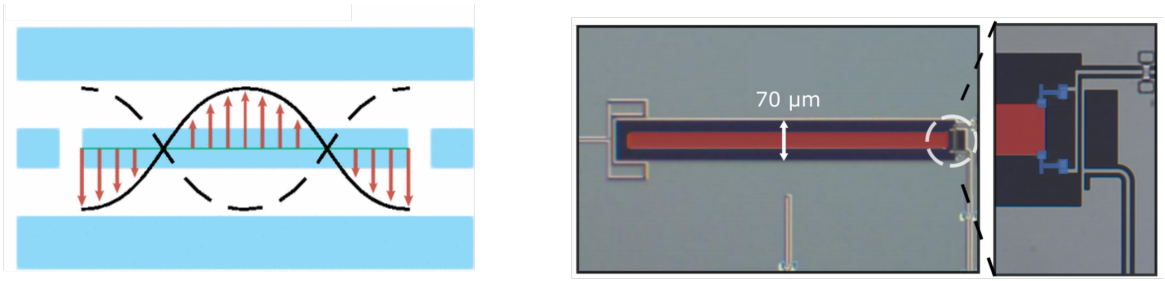


Figure 4. A spatially extended resonator, such as a long CPW, can accommodate many modes beside its fundamental. A Josephson-based qubit is a lumped-element resonator, where $|0\rangle$ and $|1\rangle$ refers to the occupancy of its lowest energy mode. It supports other oscillation modes but they are relatively high in frequency that we can safely ignore them.

What is the Cooper pair difference between the two islands? In a Josephson junction a thin oxide barrier separates two macroscopic islands. On each island there is on the order of 10^{13} electrons that pair up to form Cooper pairs. How can these two very large numbers of Cooper pairs be different only by a few and difference be stable ? Is that even meaningful to talk about such numbers ? The key concept to understand is that because of the small capacitance of the junction, the energy difference $(4\,e^2/2C)$ associated with tunneling one pair from one island to the other is rather large and $\gg k_BT$. ($C = 1$ fF gives 20 GHz and K_BT at 20 mK is 400 MHz.) Hence, the difference is stable and meaningful. While the Cooper pair number difference is small for all qubits, Trasmons are different than historic Cooper pair box qubits. For Transmons E_J/E_C =80, and as a result the wavefunction is localized is phase and spread in charge (phase is a good quantum number and charge is not). Therefore, charge fluctuations do not have significant effect. This is the essence of charge insensitivity of Transmons (**Problem 9**). For Cooper pair box qubits, charge was a good quantum number and hence it was sensitive to charge fluctuations and hard to work with.

Boltzmann factor within a mode but not across modes. Consider an oscillator that supports several modes with energies $E_{\text{mode,m}}$. Then *each mode* is a harmonic oscillator; i.e., the number of excitation in each mode (its occupancy) is given by the Boltzmann factor. For instance, the probability of finding five photons in the first mode (with energy $E_{1,5}$) is related to other photon number probabilities of the *same mode* (with energies $E_{1,m}$) by the Boltzmann factor. However, there is no Boltzmann factor associated with the probability of occupation of different modes based on the relative energies of the modes (each mode occupation is an independent variable with respect to other modes).

Where can we find photons? Now, we are in the position to provide a complete physical picture for these excitations that we use as the computational states. The probability of measuring phase difference φ at position x, $P(x,\varphi)$, has two factors. The first one is the standing-wave part, along the lines of $\cos(\pi x/L)$, and the other factor is $|\Psi(\varphi)|^2$. The computations that we did for the eigenstates of the Hamiltonian (fig. 3) only depicts the latter factor. The total probability is given by their product $P(x,\varphi) = \cos(\pi x/L)\,|\Psi(\varphi)|^2\,dx\,d\varphi$. It should be understandable that the cosine is only a placeholder for spatial extend of the wavefunction and should not be taken literally and is used to give a clear example of spatial extend.

The physical picture of $|2\rangle$ state. Does being in $|2\rangle$ state mean having two photons or one higher energy photon ? What we like to know is the plasmonic condensate oscillates. Is there a single pattern or two spatiotemporal patterns ? Since the two excitations belong to the same mode, then maybe their spatial extend is similar, but what is their temporal relation ? is it even meaningful to ask what is their relative phase ? Why the second photon decays faster ? The common answer is that ' because there are two photons there, and hence twice chance for decay'.

4. SYSTEM OF COUPLED RESONATORS

3.1. Capacitively coupled qubits

To begin with we consider two coupled linear oscillators (Fig. 5). The Lagrangian of the system is

$$\mathcal{L} = \frac{C_A}{2}\dot{\Phi}_A^2 + \frac{C_B}{2}\dot{\Phi}_B^2 + \frac{C_C}{2}(\dot{\Phi}_A - \dot{\Phi}_B)^2 - \frac{\Phi_A^2}{2L_A} - \frac{\Phi_B^2}{2L_B}. \tag{24}$$

To be sure we obtained the correct Lagrangian we look at the equations of motion

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\Phi}_A} - \frac{\partial \mathcal{L}}{\partial \Phi_A} = 0 \rightarrow \qquad C_A\ddot{\Phi}_A + C_C(\ddot{\Phi}_A - \ddot{\Phi}_B) + \frac{\Phi_A}{L_A} = 0 \tag{25}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\Phi}_B} - \frac{\partial \mathcal{L}}{\partial \Phi_B} = 0 \rightarrow \qquad C_B\ddot{\Phi}_B + C_C(\ddot{\Phi}_B - \ddot{\Phi}_A) + \frac{\Phi_B}{L_B} = 0, \tag{26}$$

which is what Kirchhoff rule tells us about current at each node.

Next, we consider the case of two coupled transmons coupled with a capacitor C_C. We use the flux at the "high" node of each qubit Φ_A and Φ_B as our canonical variable. The Lagrangian of the system can be written as

$$\mathcal{L}_{\text{Capacitive}} = \frac{C_A}{2}\dot{\Phi}_A^2 + \frac{C_B}{2}\dot{\Phi}_B^2 + \frac{C_C}{2}(\dot{\Phi}_A - \dot{\Phi}_B)^2 - E_{J,A}\cos(\frac{2\pi}{\Phi_0}\Phi_A) - E_{J,B}\cos(\frac{2\pi}{\Phi_0}\Phi_B). \tag{27}$$

The conjugate variable of fluxes, charge variable at each node can be obtained by $Q_A = \partial\mathcal{L}/\partial\dot{\Phi}_A$ and $Q_B = \partial\mathcal{L}/\partial\dot{\Phi}_B$ resulting in

$$\begin{pmatrix} Q_A \\ Q_B \end{pmatrix} = \begin{pmatrix} C_A + C_C & -C_C \\ -C_C & C_B + C_C \end{pmatrix}\begin{pmatrix} \dot{\Phi}_A \\ \dot{\Phi}_B \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} \dot{\Phi}_A \\ \dot{\Phi}_B \end{pmatrix} = \frac{1}{\text{Det}}\begin{pmatrix} C_B + C_C & C_C \\ C_C & C_A + C_C \end{pmatrix}\begin{pmatrix} Q_A \\ Q_B \end{pmatrix}, \tag{28}$$

where $\text{Det} = C_A\,C_B + C_A\,C_C + C_B\,C_C$. Using $H = \sum Q\dot{\Phi} - \mathcal{L}$, the Hamiltonian of the system can be derived

$$H_{\text{Cap}} = 4\frac{e^2}{2\widetilde{C}_A}\hat{\mathcal{N}}_A^2 + 4\frac{e^2}{2\widetilde{C}_B}\hat{\mathcal{N}}_B^2 + 4\frac{e^2}{2\widetilde{C}_C}\hat{\mathcal{N}}_A\hat{\mathcal{N}}_B - E_{J,A}\cos(\hat{\varphi}_A) - E_{J,B}\cos(\hat{\varphi}_B), \quad \text{capacitively coupled.} \tag{29}$$

$\widetilde{C}_A = C_A + C_B C_C/(C_B + C_C)$, $\widetilde{C}_B = C_B + C_A C_C/(C_A + C_C)$, and $\widetilde{C}_C = \text{Det}/\,2C_C$. Note that these relations suggest that because of coupling the effective mass of each oscillator has increased ($\widetilde{C}_A$, and $\widetilde{C}_B$ are capacitances of the nodes).

3.2. Inductively coupled qubits

The Lagrangian of the system of two inductively coupled LC oscillators is given by

$$\mathcal{L} = \frac{C_A}{2}\dot{\Phi}_A^2 + \frac{C_B}{2}\dot{\Phi}_B^2 - \frac{1}{2}\begin{bmatrix} I_A & I_B \end{bmatrix}\begin{bmatrix} \Phi_A \\ \Phi_B \end{bmatrix}, \tag{30}$$

where currents and fluxes are related by

$$\begin{bmatrix}\Phi_A\\ \Phi_B\end{bmatrix} = \begin{bmatrix}L_A & M\\ M & L_B\end{bmatrix}\begin{bmatrix}I_A\\ I_B\end{bmatrix}. \tag{31}$$

Placing Eqn. (31) in Eqn. (30) gives

$$\mathcal{L} = \frac{C_A}{2}\dot{\Phi}_A^2 + \frac{C_B}{2}\dot{\Phi}_B^2 - \frac{1}{2(L_A L_B - M^2)}\begin{bmatrix}\Phi_A & \Phi_B\end{bmatrix}\begin{bmatrix}L_B & -M\\ -M & L_A\end{bmatrix}\begin{bmatrix}\Phi_A\\ \Phi_B\end{bmatrix} = \frac{C_A}{2}\dot{\Phi}_A^2 + \frac{C_B}{2}\dot{\Phi}_B^2 - \frac{\Phi_A^2}{2\widetilde{L}_A} - \frac{\Phi_B^2}{2\widetilde{L}_B} + \frac{\Phi_A\,\Phi_B}{\widetilde{M}}, \tag{32}$$

where $\widetilde{L}_A = L_A - M^2/L_B$, $\widetilde{L}_B = L_B - M^2/L_A$, and $\widetilde{M} = L_A L_B/M - M$. The Hamiltonian of two inductively coupled transmons takes a similar form, and we leave the derivation to the readers.

For the two coupling schemes that we discussed, the interaction terms are

$$H^{\text{int}}_{\text{Cap}} \sim Q_A\, Q_B, \qquad H^{\text{int}}_{\text{Ind}} \sim \Phi_A\, \Phi_B. \tag{33}$$

Using our definitions of raising and lowering operators gives

$$H^{\text{int}}_{\text{Cap}} = J_C\,(a - a^\dagger)_A\,(a - a^\dagger)_B, \qquad H^{\text{int}}_{\text{Ind}} = J_I\,(a + a^\dagger)_A\,(a + a^\dagger)_B\,. \tag{34}$$

In the rotating frame and using the rotating wave approximation, the terms that do not conserve particle number drop to zero and for both cases we arrive at

$$H^{\text{int}}_{\text{Cap}} = J_C\,(a_A\,a_B^\dagger + a_A^\dagger\,a_B), \qquad H^{\text{int}}_{\text{Ind}} = J_I\,(a_A\,a_B^\dagger + a_A^\dagger\,a_B). \tag{35}$$

Truncating to two lowest levels gives

$$H^{\text{int}}_{\text{Cap}} \to J_C\,(\sigma_A^X\sigma_B^X + \sigma_A^Y\sigma_B^Y), \qquad H^{\text{int}}_{\text{Ind}} \to J_I\,(\sigma_A^X\sigma_B^X + \sigma_A^Y\sigma_B^Y). \tag{36}$$

We have seen in eqn. (21) that non-linearity of qubits gives the on-site interaction of the Bose-Hubbard Hamiltonian. Now, we can put all terms together and arrive at the Bose-Hubbard Hamiltonian

$$\hat{H}_{\text{Bose-Hubbard}} = \sum_j h_j a_j\,a_j^\dagger + \sum_{<i,j>} J_{i,j}(a_i\,a_j^\dagger + a_i^\dagger\,a_j) - \sum_i \frac{U_j}{2} a_j\,a_j^\dagger(a_j\,a_j^\dagger - 1). \tag{37}$$

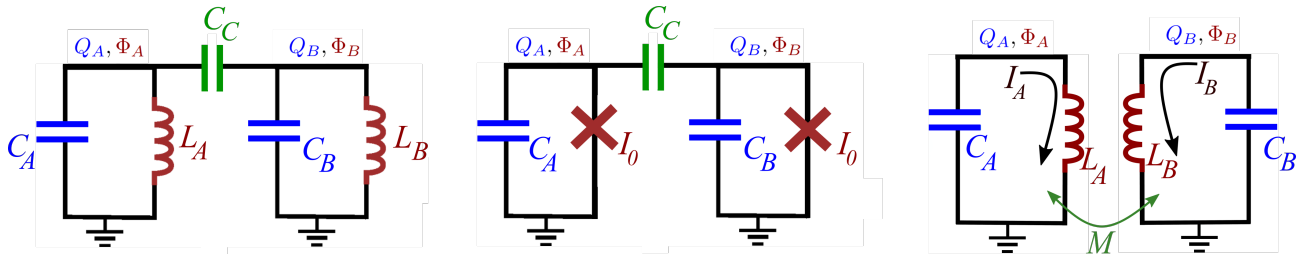


Figure 5. Schematic of two capacitively coupled LC resonators, qubits, and inductively coupled LC resonators.

Scalable approach? As we couple more oscillators, each one adds at least one degree of freedom to the problem. We are facing the numerical challenge of diagonalizing large matrices formed from tensor product of associated raising and lowering operators. One way to circumvent this challenge is to truncate to smaller raising and lowering operator for each degree of freedom which comes at the cost of reducing accuracy. The other approach is to ignore non-linear effects and solve classically as a system of coupled linear oscillators to gain some insight. The conventional electrical engineering approach models quantum circuits with lumped-elements.

We give the highlights of this approach by revisiting the case of capacitive coupling. The circuit is looked at as a two-port network, and We need to construct the admittance and impedance matrix of this two port network. These matrices can be looked up in a microwave engineering textbook, such as [2]. The inductance matrix $\mathbb{L}$ and capacitance matrix $\mathbb{C}$ are related to admittance and impedance matrix by $\mathbb{Y} = i\omega\,\mathbb{C}$ and $\mathbb{Z} = i\omega\,\mathbb{L}$. This results in

$$\mathbb{C} = \begin{bmatrix} C_A + C_C & -C_C \\ -C_C & C_B + C_C \end{bmatrix}, \qquad \mathbb{L} = \begin{bmatrix} L_A & 0 \\ 0 & L_B \end{bmatrix}. \tag{38}$$

The equation that does the magic and gives all the eigen-frequencies is

$$\boxed{\text{Det}(\mathbb{L}^{-1} - \omega^2\mathbb{C}) = 0} \tag{39}$$

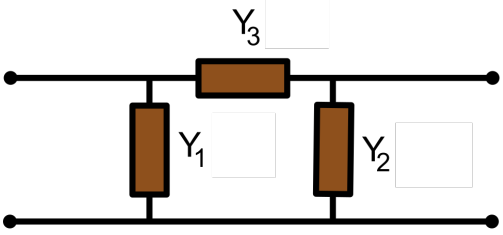
for our current case of two capacitively coupled linear oscillators, gives

$$\text{Det}(\begin{bmatrix} 1/L_A - (C_A + C_C)\,\omega^2 & C_C\,\omega^2 \\ C_C\,\omega^2 & 1/L_B - (C_B + C_C)\,\omega^2 \end{bmatrix}) = 0. \tag{40}$$

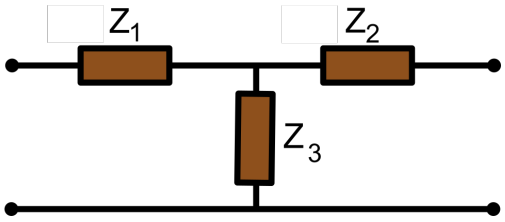
Assuming $L_A = L_B$ and $C_A = C_B$, and $\omega_0^2 \equiv 1/L_A(C_A + C_C)$, then

$$\omega = \frac{\omega_0}{\sqrt{1 \pm \dfrac{C_C}{C_C + C_A}}}. \tag{41}$$

Therefore, as the result of the coupling the two uncoupled levels are pushed apart symmetrically.

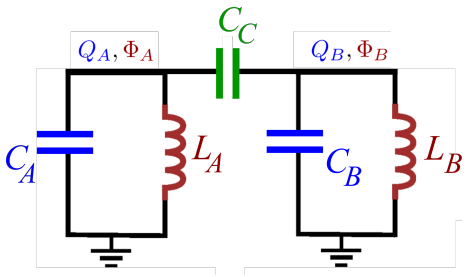


$$\mathbb{Y} = \begin{bmatrix} Y_1 + Y_3 & -Y_3 \\ -Y_3 & Y_2 + Y_3 \end{bmatrix},$$



$$\mathbb{Z} = \begin{bmatrix} Z_1 + Z_3 & Z_3 \\ Z_3 & Z_2 + Z_3 \end{bmatrix}. \tag{42}$$

5 INDUCTANCE AND CAPACITANCE MATRICES



Example 1. The Lagrangian of the system can be written

$$\mathcal{L}(\Phi, \dot{\Phi}) = \frac{C_A}{2}\dot{\Phi}_A^2 + \frac{C_B}{2}\dot{\Phi}_B^2 + \frac{C_C}{2}(\dot{\Phi}_A - \dot{\Phi}_B)^2 - \frac{\Phi_A^2}{2L_A} - \frac{\Phi_B^2}{2L_B}. \tag{43}$$

By taking the derivative of the Lagrangian, we derive the equations of motion

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\Phi}_A} - \frac{\partial \mathcal{L}}{\partial \Phi_A} = 0 \rightarrow \quad C_A\ddot{\Phi}_A + C_C(\ddot{\Phi}_A - \ddot{\Phi}_B) + \frac{\Phi_A}{L_A} = 0 \tag{44}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\Phi}_B} - \frac{\partial \mathcal{L}}{\partial \Phi_B} = 0 \rightarrow \quad C_B\ddot{\Phi}_B + C_C(\ddot{\Phi}_B - \ddot{\Phi}_A) + \frac{\Phi_B}{L_B} = 0, \tag{45}$$

which is what Kirchhoff rules provides. Considering a general decomposition for Φ_A and Φ_B in the Fourier domain

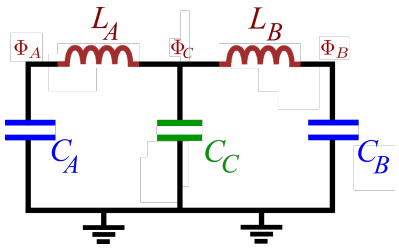
$$\Phi_A = A_1 e^{-i\omega t}, \quad \Phi_B = A_2 e^{-i\omega t} \tag{46}$$

we arrive at

$$\begin{bmatrix} (C_A + C_C)\,\omega^2 - 1/L_A & -C_C\,\omega^2 \\ -C_C\,\omega^2 & (C_B + C_C)\,\omega^2 - 1/L_B \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0. \tag{47}$$

The inductance matrix $\mathbb{L}$ and capacitance matrix $\mathbb{C}$ are

$$\mathbb{C} = \begin{bmatrix} C_A + C_C & -C_C \\ -C_C & C_B + C_C \end{bmatrix}, \qquad \mathbb{L}^{-1} = \begin{bmatrix} 1/L_A & 0 \\ 0 & 1/L_B \end{bmatrix}. \tag{48}$$



Example 2. The Lagrangian of the system can be written

$$\mathcal{L}(\Phi, \dot{\Phi}) = \frac{C_A}{2}\dot{\Phi}_A^2 + \frac{C_B}{2}\dot{\Phi}_B^2 + \frac{C_C}{2}\dot{\Phi}_C^2 - \frac{(\Phi_A - \Phi_C)^2}{2L_A} - \frac{(\Phi_B - \Phi_C)^2}{2L_B}. \tag{49}$$

By taking the derivative of the Lagrangian, we derive the equations of motion

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\Phi}_A} - \frac{\partial \mathcal{L}}{\partial \Phi_A} = 0 \rightarrow \qquad C_A\ddot{\Phi}_A + \frac{\Phi_A - \Phi_C}{L_A} = 0 \tag{50}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\Phi}_B} - \frac{\partial \mathcal{L}}{\partial \Phi_B} = 0 \rightarrow \qquad C_B\ddot{\Phi}_B + \frac{\Phi_B - \Phi_C}{L_B} = 0 \tag{51}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\Phi}_C} - \frac{\partial \mathcal{L}}{\partial \Phi_C} = 0 \rightarrow \qquad C_C\ddot{\Phi}_C + \frac{\Phi_C - \Phi_A}{L_A} + \frac{\Phi_C - \Phi_B}{L_B} = 0 \tag{52}$$

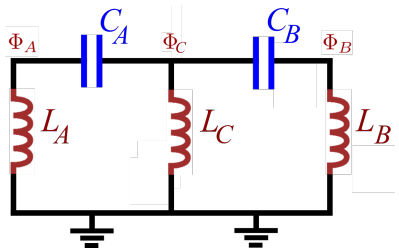
which is what Kirchhoff rules provides. Considering a general decomposition for Φ_A and Φ_B in the Fourier domain

$$\Phi_A = A_1 e^{-i\omega t}, \quad \Phi_B = A_2 e^{-i\omega t}, \quad \Phi_C = A_3 e^{-i\omega t} \tag{53}$$

The inductance matrix $\mathbb{L}$ and capacitance matrix $\mathbb{C}$ are

$$\mathbb{C} = \begin{bmatrix} C_A & 0 & 0 \\ 0 & C_C & 0 \\ 0 & 0 & C_B \end{bmatrix}, \qquad \mathbb{L}^{-1} = \begin{bmatrix} 1/L_A & -1/L_A & 0 \\ -1/L_A & 1/L_A + 1/L_B & -1/L_B \\ 0 & -1/L_B & 1/L_B \end{bmatrix}. \tag{54}$$

Where Det$(\mathbb{L}^{-1} - \omega^2\mathbb{C}) = 0$.



Example 3. The Lagrangian of the system can be written

$$\mathcal{L}(\Phi, \dot{\Phi}) = \frac{C_A}{2}(\dot{\Phi}_A - \dot{\Phi}_C)^2 + \frac{C_B}{2}(\dot{\Phi}_B - \dot{\Phi}_C)^2 - \frac{\Phi_A^2}{2L_A} - \frac{\Phi_B^2}{2L_B} - \frac{\Phi_C^2}{2L_C}. \tag{55}$$

By taking the derivative of the Lagrangian, we derive the equations of motion

$$C_A(\ddot{\Phi}_A - \ddot{\Phi}_C) + \frac{\Phi_A}{L_A} = 0, \quad C_B(\ddot{\Phi}_B - \ddot{\Phi}_C) + \frac{\Phi_B}{L_B} = 0, \quad C_A(\ddot{\Phi}_C - \ddot{\Phi}_A) + C_B(\ddot{\Phi}_C - \ddot{\Phi}_B) + \frac{\Phi_C}{L_C} = 0, \tag{56}$$

which is what Kirchhoff rules provides. Considering a general decomposition for Φ_A and Φ_B in the Fourier domain

$$\Phi_A = A_1 e^{-i\omega t}, \quad \Phi_B = A_2 e^{-i\omega t}, \quad \Phi_C = A_3 e^{-i\omega t} \tag{57}$$

The inductance matrix $\mathbb{L}$ and capacitance matrix $\mathbb{C}$ are

$$\mathbb{C} = \begin{bmatrix} C_A & -C_A & 0 \\ -C_A & C_A + C_B & -C_B \\ 0 & -C_B & C_B \end{bmatrix}, \qquad \mathbb{L}^{-1} = \begin{bmatrix} 1/L_A & 0 & 0 \\ 0 & 1/L_C & 0 \\ 0 & 0 & 1/L_B \end{bmatrix}. \tag{58}$$

Where Det$(\mathbb{L}^{-1} - \omega^2\mathbb{C}) = 0$.

6. EIGENFREQUENCIES OF COUPLED SYSTEMS

In a system of two capacitively coupled oscillators, the capacitance and inductance matrices are

$$\mathbb{C} = \begin{bmatrix} C_A + C_C & -C_C \\ -C_C & C_B + C_C \end{bmatrix}, \qquad \mathbb{L} = \begin{bmatrix} L_A & 0 \\ 0 & L_B \end{bmatrix}. \tag{59}$$

As mentioned above, the frequency of eigenmodes can be obtained from $\text{Det}(\mathbb{L}^{-1} - \omega^2 \mathbb{C}) = 0$

$$\text{Det} \begin{bmatrix} 1/L_A - (C_A + C_C)\,\omega^2 & C_C\,\omega^2 \\ C_C\,\omega^2 & 1/L_B - (C_B + C_C)\,\omega^2 \end{bmatrix} = 0. \tag{60}$$

Diving the first row by C_A and the second row by C_B gives

$$\text{Det} \begin{bmatrix} \omega_A^2 - (1 + \kappa_A)\,\omega^2 & \kappa_A\,\omega^2 \\ \kappa_B\,\omega^2 & \omega_B^2 - (1 + \kappa_B)\,\omega^2 \end{bmatrix} = 0, \tag{61}$$

where $\kappa_A \equiv C_C/C_A$, $\kappa_B \equiv C_C/C_B$. For simplicity, let's assume $\kappa_A = \kappa_B = \kappa$, and $\omega_A = \omega_B = \omega_0$, then

$$\omega_1 = \omega_0, \quad \omega_2 = \frac{\omega_0}{\sqrt{1 + 2\kappa}} \longrightarrow \omega_0\,(1 - \kappa) \tag{62}$$

Note that if we have defined $\widetilde{C}_A \equiv C_A + C_C$, $\widetilde{C}_B \equiv C_B + C_C$, $\widetilde{\kappa_A} \equiv C_C/\widetilde{C}_A$, and $\widetilde{\omega_0}^2 \equiv L_A \widetilde{C}_A$

$$\omega_{1,2} = \frac{\widetilde{\omega_0}}{\sqrt{1 \pm \dfrac{C_C}{\sqrt{\widetilde{C}_A \widetilde{C}_B}}}} \longrightarrow \widetilde{\omega_0}\,(1 \pm \frac{C_C}{2\sqrt{\widetilde{C}_A \widetilde{C}_B}}) = \widetilde{\omega_0}\,(1 \pm \widetilde{\kappa}/2) \tag{63}$$

Which one is more physical? C_A or $\widetilde{C}_A$?

The eigenenergies can also be obtained from Hamiltonian $H_{QA} + H_{QB} - g(a - a^\dagger)_A(a - a^\dagger)_B$

$$\begin{bmatrix} f_{Q1} & -\kappa/2\sqrt{f_{Q1}\,f_{Q2}} \\ -\kappa/2\sqrt{f_{Q1}\,f_{Q2}} & f_{Q2} \end{bmatrix} \longrightarrow eigs = f_Q\,(1 \pm \kappa/2), \tag{64}$$

where $\kappa \equiv C_C/\sqrt{\widetilde{C}_A \widetilde{C}_B}$. Note that although both approaches give the same result in small coupling, the equations that use are different. one has ω^2 next to each element, and the other subtracts ω from diagonal. Same results but different equations and operations. More generally the eigenvalues of Eqn.(6) are given by

$$eigs = \frac{f_{Q1} + f_{Q2}}{2} \pm \sqrt{\left|\frac{f_{Q1} - f_{Q2}}{2}\right|^2 + g^2} \tag{65}$$

where for convenience we have defined $g \equiv \kappa\sqrt{f_{Q1}\,f_{Q2}}/2$

Next, we look at three capacitively coupled oscillators [3]. The eigen-frequencies of the system can be obtained from

$$\text{Det} \begin{bmatrix} \frac{1}{L_q} - \omega^2(C_q + C_{qC} + C_{qq}) & \omega^2 C_{qC} & \omega^2 C_{qq} \\ \omega^2 C_{qC} & \frac{1}{L_C} - \omega^2(C_C + 2C_{qC}) & \omega^2 C_{qC} \\ \omega^2 C_{qq} & \omega^2 C_{qC} & \frac{1}{L_q} - \omega^2(C_q + C_{qC} + C_{qq}) \end{bmatrix} = 0. \tag{66}$$

We solve this by splitting the matrix into direct and indirect coupling matrices [3], and define coupling as half of the splitting:

$$\text{Total coupling:} \quad g = \left[\gamma_{tot} - \kappa^2 \frac{\omega_1\omega_2}{\omega_{cp}^2 - \omega_1\omega_2}\right] \frac{\sqrt{\omega_1\omega_2}}{2}. \tag{67}$$

For completeness, we write the Hamiltonian of the system:

$$H = \left[a^\dagger a + 1/2\right]_{Q1} \omega_1 + \left[a^\dagger a + 1/2\right]_{Q2} \omega_2 + \left[a^\dagger a + 1/2\right]_{CP} \omega_{cp} + H_{\text{couplers}}, \tag{68}$$

with

$$H_{\text{couplers}} = -\kappa \frac{\sqrt{\omega_1\omega_{cp}}}{2} \left[a - a^\dagger\right]_{Q1} \left[a - a^\dagger\right]_{CP} - \kappa \frac{\sqrt{\omega_2\omega_{cp}}}{2} \left[a - a^\dagger\right]_{Q2} \left[a - a^\dagger\right]_{CP} - \gamma \frac{\sqrt{\omega_1\omega_2}}{2} \left[a - a^\dagger\right]_{Q1} \left[a - a^\dagger\right]_{Q2} \tag{69}$$

Note that charge-charge coupling, coming from capacitive coupling, has a minus sign coming from $i = \sqrt{-1}$ in the definition of charge.We can truncate the operators and write the Hamiltonian in the basis of the uncoupled harmonic oscillators. To the lowest order

$$\begin{bmatrix} \omega_1 & \kappa/2\sqrt{\omega_1\,\omega_{cp}} & \gamma/2\sqrt{\omega_1\,\omega_2} \\ \kappa/2\sqrt{\omega_1\,\omega_{cp}} & \omega_{cp} & \kappa/2\sqrt{\omega_2\,\omega_{cp}} \\ \gamma/2\sqrt{\omega_1\,\omega_2} & \kappa/2\sqrt{\omega_2\,\omega_{cp}} & \omega_2 \end{bmatrix}, \tag{70}$$

Note that the sign of the coupling matrix elements are all positive, since $\langle 01|(a-a^\dagger)\otimes(a-a^\dagger)|10\rangle = -1$. Although this truncated version can show properties such as cancellation of direct (positive g) and indirect (negative g) couplings, it is missing correction factors that cannot be ignored. Note that solving Eqn.(10), is equivalent to solving $eigs(H)$ without truncation. Therefore, in solving linear coupled harmonic oscillators, writing Hamiltonian is not beneficial, since one needs to solve by truncation which introduce errors; yet Eqn.(10) provides exact. Unless we are after non-linearity corrections, there is no need to write and solve the Hamiltonian.

At the level of two qubits, the requested Hamiltonian is

$$\begin{bmatrix} \delta_1 & J_{12} \\ J_{12} & \delta_2 \end{bmatrix}, \tag{71}$$

where δ_1 and $\delta_2 \in [-\Delta, \Delta]$. We bring qubits to $f_{ave} = (f_{Q1}^{10} + f_{Q2}^{10})/2$, which we call interaction place and expect to see peaks at the eigenvalues of

$$\begin{bmatrix} \delta_1 + f_{ave} & J_{12} \\ J_{12} & \delta_2 + f_{ave} \end{bmatrix}. \tag{72}$$

Note that this is in absolute frequency scale, which is indeed what we can measure in the lab. Now we ask how to construct a Hamiltonian. Find amplitudes that move qubits to f_{Q1} and f_{Q2} frequencies

$$f_{Q1} = f_{ave} + \delta_1 + J_{12} \qquad f_{Q2} = f_{ave} + \delta_2 + J_{12}, \tag{73}$$

and find amplitudes that moves the coupler to f_{cp} that satisfies this relation

$$J_{12} = \left(\gamma - \frac{\kappa^2}{(f_{cp}/f_Q)^2 - 1}\right) \frac{f_Q}{2}, \tag{74}$$

where $f_Q = \sqrt{f_{Q1} f_{Q2}}$.

7. THE ROTATING FRAME PICTURE

Many observables in a superconducting circuits are measurement in the so-called rotating frame. Is it just a convenient choice or a necessity? Let's use a few examples to review the transformation to the rotating frame. Consider the time-independent operator R_z that rotates a single qubit *state* by θ in the x-y plane,

$$R_z = \exp(-i\frac{\theta}{2}Z) = \begin{bmatrix} \exp(-i\,\theta/2) & 0 \\ 0 & \exp(i\,\theta/2) \end{bmatrix}. \tag{75}$$

This change of frame is merely a change of view point, therefore to assure the properties of the state remains the same, we are interested to know how the observables that describe a given state transform. For example, for stabilizer states, how the set of stabilizers transform to the new basis. Stabilizers transform according to

$$S_{j,\,\text{new frame}} = R_z\, S_{j,\,\text{old frame}}\, R_z^\dagger \tag{76}$$

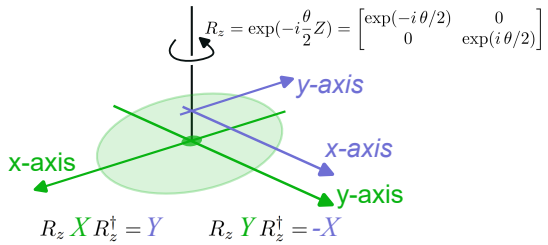
Exercise. Show that when $\theta = \pi/2$, the $\hat{X}$ operator (stabilizer) in the new frame becomes $\hat{Y}$, and the $\hat{Y}$ operator (stabilizer) becomes $-\hat{X}$.

In appendix A we discuss how a Hamiltonian H_S transform under time-dependent frame change to H_I ? As the simplest example, consider $H_S = \omega_0/2\, Z$ and a frame that rotates around z-axis with rate ω_D,

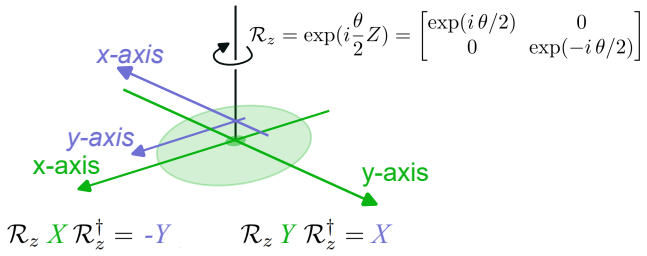
$$U = \exp(-i\frac{\omega_D t}{2}Z) = \begin{bmatrix} \exp(-i\frac{\omega_D t}{2}) & 0 \\ 0 & \exp(i\frac{\omega_D t}{2}) \end{bmatrix} \quad \rightarrow \quad \dot{U} = \begin{bmatrix} -i\frac{\omega_D}{2}\exp(-i\frac{\omega_D}{2}) & 0 \\ 0 & i\frac{\omega_D}{2}\exp(i\frac{\omega_D}{2}) \end{bmatrix} \tag{77}$$

$$H_I = U^\dagger H_S U + i\dot{U}^\dagger U = H_S + i\dot{U}^\dagger U = \begin{bmatrix} \omega_0/2 & 0 \\ 0 & -\omega_0/2 \end{bmatrix} + \begin{bmatrix} -\omega_D/2 & 0 \\ 0 & \omega_D/2 \end{bmatrix} = \begin{bmatrix} (\omega_0-\omega_D)/2 & 0 \\ 0 & -(\omega_0-\omega_D)/2 \end{bmatrix} \tag{78}$$

How stabilizers transform



How Hamiltonian terms transform



Consider a 2-level system, with levels $|0\rangle$ and $|1\rangle$, and initial state $|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$. The Hamiltonian of the system is $\mathcal{H}_{\text{qubit}}$ and and the state of the system at time t is $|\psi(t)\rangle$,

$$\mathcal{H}_{\text{qubit}} = \omega_g|0\rangle\langle 0| + \omega_e|1\rangle\langle 1|, \qquad |\psi(t)\rangle = \alpha e^{-i\omega_g t}|0\rangle + \beta e^{-i\omega_e t}|1\rangle. \tag{79}$$

The expectation values of $\hat{X}$ and $\hat{Z}$, in the stationary frame is

$$\langle X\rangle = 2\,\mathsf{Re}(\alpha\overline{\beta})\,\cos[(\omega_e-\omega_g)t],\quad \langle Z\rangle = \alpha\overline{\alpha}-\beta\overline{\beta}. \tag{80}$$

Therefore, the rotation naturally emerges as a result of energy differences. If we chose to go to a frame that rotates with $\omega_e-\omega_g$ frequency, then we could turn this into a stationary state.

Next, consider a 3-level system, with levels $|0\rangle$, $|1\rangle$ and $|2\rangle$, with initial state $|\psi_0\rangle = \alpha|0\rangle+\beta|1\rangle+\gamma|2\rangle$. The Hamiltonian of the system is

$$\mathcal{H}_{\text{qutrit}} = \omega_g|0\rangle\langle 0| + \omega_e|1\rangle\langle 1| + \omega_f|2\rangle\langle 2|. \tag{81}$$

and the state of the system at time t is

$$|\psi(t)\rangle = \alpha e^{-i\omega_g t}|0\rangle + \beta e^{-i\omega_e t}|1\rangle + \gamma e^{-i\omega_f t}|2\rangle. \tag{82}$$

Based on the commonly used parameters, the energy of $|2\rangle$ state with respect $|0\rangle$ state is almost twice of that of the $|1\rangle$ state, $\omega_e-\omega_g\sim 6\,$GHz and $\omega_f-\omega_g\sim 11.7\,$GHz. The key concept to notice is how Pauli X or the Rabi drive is defined.

$$X_{|1\rangle-|2\rangle} = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix},\ \rightarrow\ \langle X_{|1\rangle-|2\rangle}\rangle = \alpha\overline{\alpha}+2\,\mathsf{Re}(\beta\,\overline{\gamma})\,\cos[(\omega_f-\omega_e)t]. \tag{83}$$

Therefore, it is the frequency difference that enters this equation that sets the clock rate for exciting and to $|2\rangle$ state and its reading. So, no equipment to operate at 12 GHz are needed.

8. THE HIGHER ORDER TERMS IN PAULI EXPANSION (ARRIVING SOON!)

9. SINGLE-PHOTON DYNAMICAL SPECTROSCOPY TECHNIQUE

In this section we provide a pedagogical illustration of the basics of multi-photon spectroscopy by stepping through two simple examples. We focus on Hamiltonian dynamics, which is adequate for illustrating the technique. The Fourier transform in space that was done in Fig. 7 is resulting from the space invariance of the circuit. For simplicity we only focus on the time-energy transformation and do not discuss the space-momentum transformation.

Consider a Hamiltonian $\hat{\mathcal{H}}$ with eigenenergies ω_n and eigenstates φ_n

$$\hat{\mathcal{H}}\varphi_n = \omega_n \varphi_n. \tag{84}$$

We seek a dynamical approach to learn the spectrum of $\mathcal{H}$. Consider an initial state ψ_0 and its evolution ψ_t

$$\psi_0 = \sum_n c_n \varphi_n, \qquad \psi_t = e^{-i\hat{\mathcal{H}}t}\psi_0 = \sum_n c_n\, e^{-i\omega_n t}\varphi_n, \tag{85}$$

where the complex coefficient $c_n = \varphi_n \psi_0\rangle$ is the overlap of ψ_0 with φ_n. The desired spectrum of $\hat{\mathcal{H}}$ can be extracted (e.g. by Fourier transform) from the overlap of ψ_0 and ψ_t

$$\langle\psi_0|\psi_t\rangle = \sum_n |c_n|^2\, e^{-i\omega_n t}\ . \tag{86}$$

This overlap also frequently appears in non-equilibrium dynamics questions.

Consider an initial state $|\Psi_0\rangle$, which evolves under Hamiltonian H to $|\Psi_t\rangle = e^{-iHt}|\Psi_0\rangle$, and a Hermitian operator O at site m that we like to measure it at time t,

$$\langle\Psi_t|\hat{O}_m|\Psi_t\rangle = \langle\Psi_0|\, e^{iHt}\, \hat{O}_m\, e^{-iHt}\, |\Psi_0\rangle = \langle\Psi_0|\, e^{iHt}\, \hat{O}_m\, |\Psi_t\rangle \tag{87}$$

which seems neither here nor there as far as having something that gives overlap of final and initial states is concerned. Now consider an initial state that is sum of two states, in two decoupled sectors that can be joined by $\hat{O}_m$, one of the sectors has one state in it, meaning that there is no evolution in it. For example, when H is excitation number conserving, and

$$|\Psi_0\rangle = |000...\rangle + |100...\rangle = |vac\rangle + |\psi_0\rangle, \quad \rightarrow \quad |\Psi_t\rangle = e^{-iHt}|\Psi_0\rangle = |vac\rangle + e^{-iHt}|\psi_0\rangle = |vac\rangle + |\psi_t\rangle \tag{88}$$

Then

$$C_x(m,t) = \langle\Psi_0|\, e^{iHt}\, \hat{O}_m\, e^{-iHt}\, |\Psi_0\rangle = \langle vac+\psi_0|\, e^{iHt}\, \hat{O}_m\, e^{-iHt}\, |vac+\psi_0\rangle \tag{89}$$

$$= \langle vac|\, e^{iHt}\, \hat{O}_m\, e^{-iHt}\, |\psi_0\rangle + \langle\psi_0|\, e^{iHt}\, \hat{O}_m\, e^{-iHt}\, |vac\rangle \tag{90}$$

$$= \langle vac|\hat{O}_m\, e^{-iHt}\, |\psi_0\rangle + \langle\psi_0|\, e^{iHt}\, \hat{O}_m|vac\rangle, \tag{91}$$

since from 4 possible terms only two are non-zero, since $\hat{O}_m$ takes from one sector to the other, e.g $O_m = X_m$

$$= \langle\psi_0|\, e^{-iHt}\, |\psi_0\rangle + \langle\psi_0|\, e^{iHt}\, |\psi_0\rangle = \langle\psi_0|\psi_t\rangle + \langle\psi_t|\psi_0\rangle. \tag{92}$$

If we set $O_m = Y_m$, then

$$C_y(m,t) = -i\langle\psi_0|\psi_t\rangle + i\langle\psi_t|\psi_0\rangle. \tag{93}$$

and thus

$$C_x(m,t) + iC_y(m,t) = 2\langle\psi_0|\psi_t\rangle. \tag{94}$$

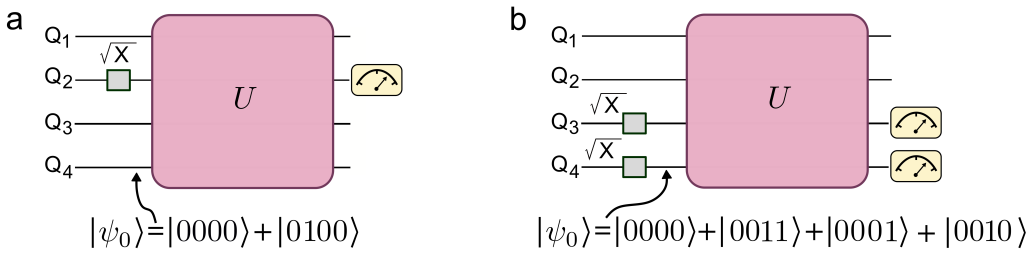


Figure 6. Pulse sequences for determining the spectra of **a,** a single excitation, and **b,** two excitations for 4 qubits.

10. HOMEWORKS

Problem 1, Josephson relation. Consider two superconductors separated by a thin insulator, such that Cooper pairs can coherently tunnel between them with rate Γ. Assume the wave-function of the condensate takes the form of $\psi_1 = \sqrt{n_1}\, e^{-i\varphi_1}$, and $\psi_2 = \sqrt{n_2}\, e^{-i\varphi_2}$, where $\sqrt{n_i}$ is the amplitude and φ_i is the phase of the Ginsburg-Landau wavefunction. The wave-functions are related by $i\,\partial\psi_1/\partial t = \Gamma\psi_2$ and $i\,\partial\psi_2/\partial t = \Gamma\psi_1$. Using these relations, derive the Josephson current relation $I = I_0 \sin(\varphi)$.

Problem 2, Transmon Hamiltonian. Given that $\eta \ll 1$, show that $\hat{H}_{\text{Qubit}}/\hbar\omega_p = 4\eta\hat{\mathcal{N}}^2 - \dfrac{\cos(\hat{\varphi})}{8\eta}$, can be re-written in terms of a and $a^\dagger$

$$H/\hbar\omega_p = a^\dagger a + 1/2 - \frac{\eta}{3}\Big(\frac{a+a^\dagger}{\sqrt{2}}\Big)^4 + \frac{4\eta^2}{45}\Big(\frac{a+a^\dagger}{\sqrt{2}}\Big)^6. \tag{95}$$

where

$$\hat{\varphi} = 2\sqrt{\eta}(a+a^\dagger), \quad \hat{\mathcal{N}} = \frac{a-a^\dagger}{4i\sqrt{\eta}}. \tag{96}$$

Problem 3, Transmon energies. Starting from Eqn. (95) and using perturbation expansion, show that the few lowest energy levels of Transmon are

$$\begin{aligned}
\omega_g &= \omega_p(\frac{1}{2} - \frac{1}{4}\eta - \frac{1}{8}\eta^2),\\
\omega_e &= \omega_p(\frac{3}{2} - \frac{5}{4}\eta - \frac{9}{8}\eta^2),\\
\omega_f &= \omega_p(\frac{5}{2} - \frac{13}{4}\eta - \frac{35}{8}\eta^2),\\
\omega_h &= \omega_p(\frac{7}{2} - \frac{25}{4}\eta - \frac{91}{8}\eta^2).
\end{aligned} \tag{97}$$

Hint: one needs to compute expectations values such as $\langle 0|\,(a+a^\dagger)^4\,|0\rangle = 3$, $\langle 1|\,(a+a^\dagger)^4\,|1\rangle = 15$, and $\langle 2|\,(a+a^\dagger)^4\,|2\rangle = 39$, where $|0\rangle,\ |1\rangle,\ |2\rangle$ are unperturbed harmonic potential states.

Problem 4, Numerical approach. If you are not interested in perturbation expansion, by numerically finding the eigen-energies of Hamiltonian shown in Problem 2, show that the relations in Eqn. (97) provide a good fit for small η values. Typical values to use are $\omega_p/2\pi$ =6 GHz, and η=0.04.

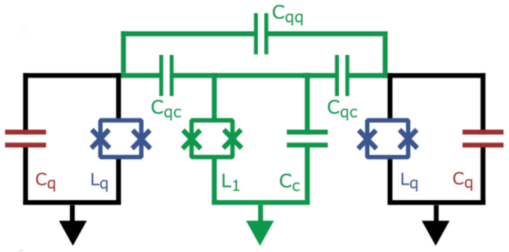
Problem 5, Eigenstates of non-linear resonators. Noting that $\hat{\mathcal{N}} = \dfrac{1}{i}\dfrac{\partial}{\partial\hat{\varphi}}$, show that

$$\hat{H}_{\text{Qubit}}/\hbar\omega_p = -4\eta\,\frac{\partial^2}{\partial\hat{\varphi}^2} - \frac{\cos(\hat{\varphi})}{8\eta}, \tag{98}$$

suitable for setting a numerical difference equation to compute eigenstates ψ in phase-space (indexed by j)

$$\hat{H}_j \rightarrow -\frac{4\eta}{(\delta\varphi)^2}\left(\psi_{j+1}-2\psi_j+\psi_{j-1}\right)-\frac{\cos\varphi}{8\eta}\,\psi_j \quad \longrightarrow \quad \left[2-\frac{(\delta\varphi)^2}{32\eta^2}\right]\psi_j-\left(\psi_{j+1}+\psi_{j-1}\right). \tag{99}$$

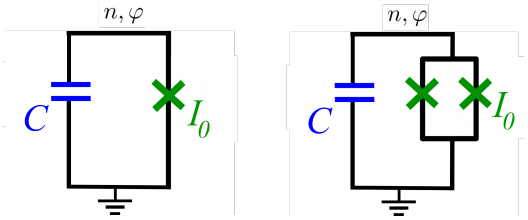
Problem 6, Three coupled oscillators. By considering a system of three capacitively coupled oscillators as a 3-port network, write the system of equations that give the eigen-frequencies of the system (see figure).



Answer:

$$\text{Det}\begin{bmatrix} \frac{1}{L_q}-\omega^2(C_q+C_{qC}+C_{qq}) & \omega^2 C_{qC} & \omega^2 C_{qq} \\ \omega^2 C_{qC} & \frac{1}{L_q}-\omega^2(C_C+2C_{qC}) & \omega^2 C_{qC} \\ \omega^2 C_{qq} & \omega^2 C_{qC} & \frac{1}{L_q}-\omega^2(C_q+C_{qC}+C_{qq}) \end{bmatrix}=0 \tag{100}$$

Problem 7, Frequency tunable qubits. Commonly, in transmons two junctions are used instead of one to form a SQUID and make a tunable frequency qubit. Show that $E_J \equiv \Phi_0\, I_0/2\pi$ need to be replaced by $2E_J\cos(\phi_{\text{ext}})$, where ϕ_{ext} is the external flux applied to the SQUID loop. Obtain eigen-energies as a function of this external flux.

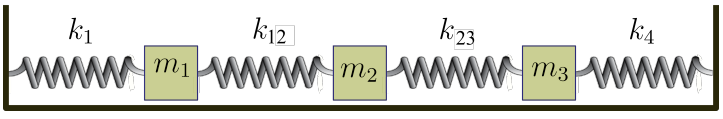


Problem 8, Physical picture.Can we distinguish 5 photons in the 1^{st} mode ($5\hbar\omega_1$) of a cavity and one photon in the 5 th mode $\hbar(5\omega_1)$?

Problem 9, Phase and charge fluctuations. Using perturbation expansion or the numerical code, find the $<\hat{n}^2>$ and $<\hat{\varphi}^2>$ for the groundstate and first excited state of a transmon with $\omega_p/2\pi$=6 GHz and $\eta = 0.04$.

Answer: for the groundstate, $<Q^2>=\hbar/2\sqrt{C/L}$ and $<\Phi^2>=\hbar/2\sqrt{L/C}$.

Problem 10, Lumped-element approximation. Consider a system of masses and springs. Assume some reasonable values for parameters and numerically solve for the eigen-modes of the system.



Notice that in the lowest energy mode of the system, i.e. the fundamental mode, the masses move together. In higher modes there is a relative motion between centers of masses. Next, assume $k_1 \ll k_{12} = k_{23}$, $m_1 = m_2 = 0$, and $k_4 = 0$. Note the change in eigen-energies. Studying this limiting case can help you to see the origin of lumped-element approximation.

Answer:

$$m\ddot{x}_j + k_{j,j-1}(x_j - x_{j-1}) - k_{j,j+1}(x_{j+1} - x_j) = 0$$

$$|\mathbb{K} - \omega^2 \mathbb{M}| = 0 \tag{101}$$

$$\mathbb{K} = \begin{bmatrix} k_1 + k_{12} & -k_{12} & 0 \\ -k_{12} & k_{12} + k_{23} & -k_{23} \\ 0 & -k_{23} & k_{23} + k_4 \end{bmatrix} = 0, \quad \mathbb{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = 0 \tag{102}$$

Although it is common to say that " capacitance is similar to mass", since there is not obvious connection between spring constant and inductance (Det$(\mathbb{L}^{-1} - \omega^2 \mathbb{C}) = 0$), therefore, this analogy does not provide much intuition.

Problem 11, Multi-excitation example. Will add soon !

APPENDIX A. BRIEF REVIEW OF HAMILTONIAN PHYSICS

In view of Hamiltonian principle's wide range applicability (even though this is an after-the-fact discovery), it is not unreasonable to assert that this principle is more "fundamental" than Newton's equations. Hamilton's Principle: *Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies*,

$$\delta \int_{t_1}^{t_2} (T-U)\, dt = 0\,. \tag{103}$$

Since Lagrangian is defined $\mathcal{L} = T - U$, the Euler-Lagrange equations for general coordinate q_i is

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0 \tag{104}$$

and the Hamiltonian of the system is given after obtaining the generalized momenta $p_j = \dfrac{\partial \mathcal{L}}{\partial \dot{q}_j}$

$$H = \sum_j p_j\, \dot{q}_j - \mathcal{L} \tag{105}$$

Newton's equations and Hamilton's Principle, and the resulting Lagrangian equations, have been shown to be entirely equivalent. Hence, no distinction exists between these viewpoints, which are based on the description of physical effects. But from a philosophical standpoint, one can make a distinction. In the Newtonian formulation, a certain force on a body produces a definite motion-that is, we always associate a definite *effect* with a certain *cause*. According to Hamilton's Principle, however, the motion of a body results from the attempt of nature to achieve a certain minimization *purpose*.

Example 1. A simple pendulum of length l and bob with mass m is attached to a massless fixed support. Find the period for small oscillations.

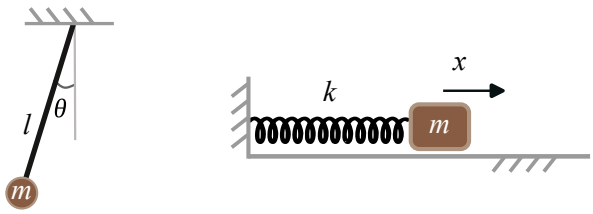
$$\mathcal{L}(\theta, \dot{\theta}) = T - U = \frac{m}{2}(l\,\dot{\theta})^2 - mgl[1-\cos(\theta)] \quad \rightarrow \quad \ddot{\theta} + \frac{g}{l}\sin(\theta) = 0 \tag{106}$$

$$P = \frac{\partial}{\partial \dot{\theta}}\mathcal{L} = m\,l^2\,\dot{\theta} \quad \rightarrow \quad H = P\,\dot{\theta} - \mathcal{L} = \frac{m}{2}(l\,\dot{\theta})^2 + mgl[1-\cos(\theta)] \tag{107}$$

Example 2. A cube of mass m is attached to a massless spring of constant k . Find the period for small oscillations.

$$\mathcal{L} = T - U = \frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2 \quad \rightarrow \quad \ddot{x} + \frac{k}{m}x = 0 \tag{108}$$

$$P = \frac{\partial}{\partial \dot{\theta}}\mathcal{L} = m\,\dot{x} \quad \rightarrow \quad H = P\,\dot{x} - \mathcal{L} = \frac{m}{2}\dot{x}^2 + \frac{k}{2}x^2 \tag{109}$$



APPENDIX B. THE SCHRÖDINGER AND HEISENBERG PICTURE

There are several equivalent descriptions of quantum dynamics. In the Schrödinger picture the state of the system is described by a vector $|\psi_S\rangle$, and the time evolution of it is given by

$$i\hbar \frac{d}{dt}|\psi_S\rangle = H_S\,|\psi_S\rangle \qquad \rightarrow \qquad |\,\psi_S(t)\,\rangle = U_S\,|\,\psi_S(0)\,\rangle. \tag{110}$$

where we define a propagator U_S. Generally our goal for going to a rotating frame or interaction picture is to cancel some part of the evolution by "unwinding" the quantum state. The Heisenberg picture is a specific interaction picture, where this unwinding is such that the quantum state evolution is completely frozen and only the operators evolve. Consider a Hamiltonian H_S and switching to a frame that rotates at the rate set by the associated unitary, U_S. To neutralize the rotation induced by U_S, we want to unwind the quantum state evolution.

$$|\,\psi_I(t)\,\rangle = U_S^\dagger\,|\,\psi_S(t)\,\rangle. \tag{111}$$

where $|\,\psi_I(t)\,\rangle$ is the state vector in the rotating frame (i.e. the interacting picture). To arrive at the Hamiltonian H_I in this frame,

$$-\,i\hbar\frac{d}{dt}|\,\psi_S\,\rangle = H_S|\,\psi_S\,\rangle \qquad \rightarrow \qquad \frac{d}{dt}|\,\psi_S\,\rangle = -\frac{i}{\hbar}H_S|\,\psi_S\,\rangle = -\frac{i}{\hbar}H_S\,U_S\,U_S^\dagger|\,\psi_S\,\rangle. \tag{112}$$

Note that $\dfrac{d}{dt}\Big(U_S^\dagger|\,\psi_S\,\rangle\Big) = \dot{U}_S^\dagger\,|\,\psi_S\,\rangle\; + U_S^\dagger\,\dfrac{d}{dt}|\,\psi_S\,\rangle$ which follow

$$\frac{d}{dt}\big(\,U_S^\dagger\,|\,\psi_S\,\rangle\,\big) = -\frac{i}{\hbar}U_S^\dagger H_S\,U_S\;U_S^\dagger\frac{d}{dt}|\,\psi_S\,\rangle + \dot{U}_S^\dagger\,U_S\;U_S^\dagger\frac{d}{dt}|\,\psi_S\,\rangle \tag{113}$$

Since $|\,\psi_I(t)\,\rangle = U_S^\dagger\,|\,\psi_S(t)\,\rangle$, the above equation gives

$$\boxed{\frac{d}{dt}|\,\psi_I\,\rangle = -\frac{i}{\hbar}H_I|\,\psi_I\,\rangle, \quad \text{where} \quad H_I = U_S^\dagger\,H_S\,U_S + i\hbar\,\dot{U}_S^\dagger\,U_S} \tag{114}$$

Note that in many references the place of U and $U^\dagger$ is changed. To be consistent, we chose U to be forward evolution in time and $U^\dagger$ being the backward evolution, which unwinds the quantum state back to time=0.

For additional reading, please see references [3–8].

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